Approximation and Coding with Orthogonal Decompositions

Gabriel Peyré

http://www.ceremade.dauphine.fr/~peyre/numerical-tour/

Overview

- Approximation and Compression
- Decay of Approximation Error
- Fourier for Smooth Functions
- Wavelet for Piecewise Smooth Functions
- Curvelets and Finite Elements for Cartoons

Sparse Approximation in a Basis

Orthogonal basis $\mathcal{B} = \{\psi_m\}_m$ of $\mathrm{L}^2([0,1]^d)$ $d = 1$ (signals) or $d = 2$ (images)

$$f = \sum_m \langle f,\, \psi_m \rangle \psi_m \xrightarrow{\text{approximation}} f_M = \sum_{m \in I_M} \langle f,\, \psi_m \rangle \psi_m$$

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Linear approximation:

I_M does not depend on f.

e.g.: $I_M = \{M$ low frequencies$\}$.

Image f

Linear approximation

Sparse Approximation in a Basis

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Linear approximation:

I_M does not depend on f.

e.g.: $I_M = \{M \text{ low frequencies}\}$.

Image f

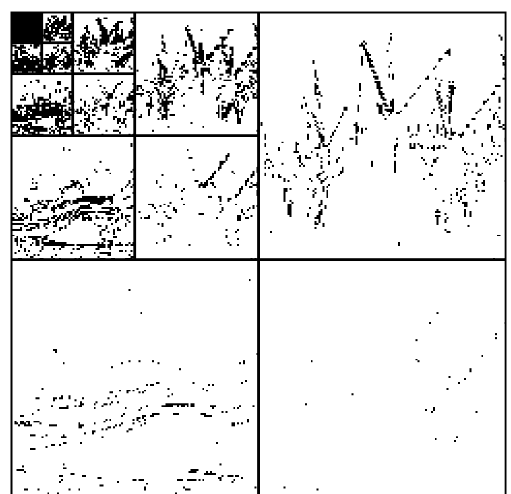
Linear approximation

Non-linear approximation:

minimize $\|f - f_M\|$ for a given M.

$$I_M = \{m \setminus |\langle f, \psi_m \rangle| > T\}$$

and $\quad M = \# I_M$.



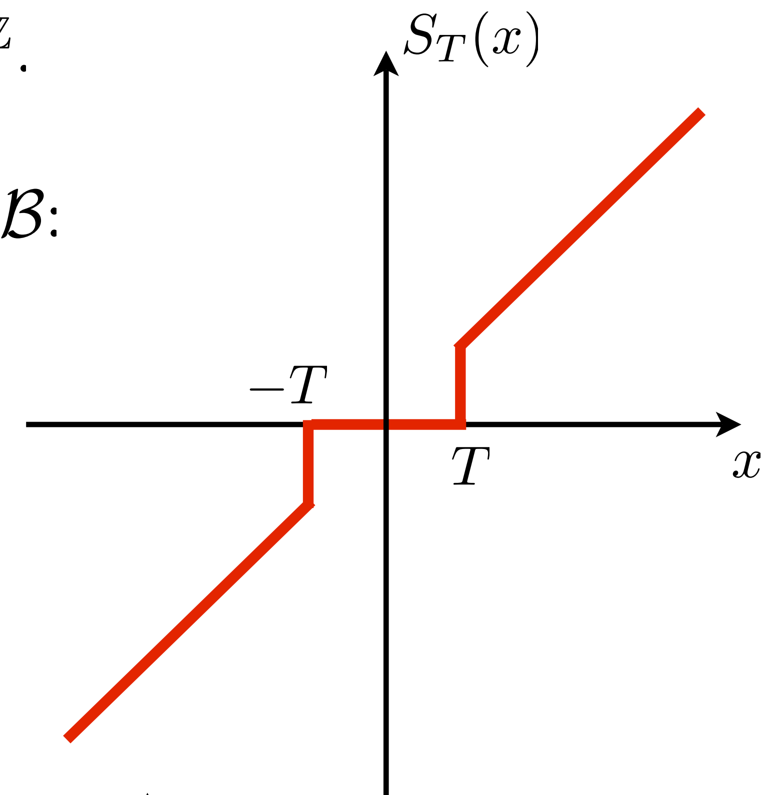
Coefficients I_T

Non-linear approximation

Hard Thresholding

Orthogonal basis $\mathcal{B} = \{\psi_m\}_{m\in\mathbb{Z}}$ of $\mathrm{L}^2([0,1]^d)$.

Transform: $f \in \mathrm{L}^2([0,1]^d) \mapsto \{\langle f,\, \psi_m\rangle\}_m \in \mathbb{R}^{\mathbb{Z}}$.



Best M-term (non-linear) approximation in $\mathcal{B}$:

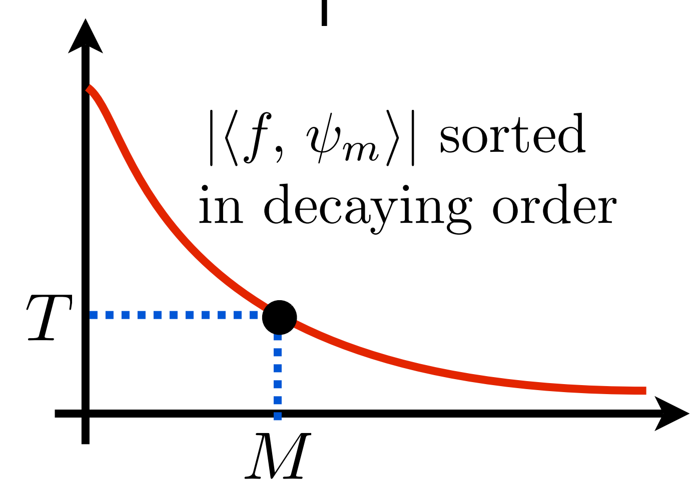
$$f_M = \sum_{|\langle f,\, \psi_m\rangle| > T} \langle f,\, \psi_m\rangle \psi_m$$

$$= \sum_m S_T(\langle f,\, \psi_m\rangle)\psi_m$$

Thresholding: $S_T(x) = \begin{cases} x & \text{if} \quad |x| > T \\ 0 & \text{if} \quad |x| \leqslant T \end{cases}$

$\#\text{coefs.} = M = \#\left\{m \setminus |\langle f,\, \psi_m\rangle| > T\right\}$

1:1 mapping $M \leftrightarrow T$



Compression by Transform-coding

$$f \xrightarrow[\text{transform}]{\text{forward}} a[m] = \langle f,\, \psi_m \rangle \in \mathbb{R}$$

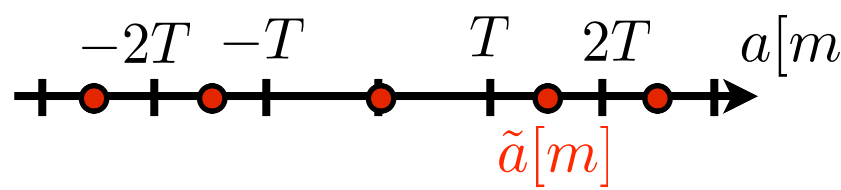
Image f

Zoom on f

Compression by Transform-coding

$$f \xrightarrow[\text{transform}]{\text{forward}} a[m] = \langle f,\, \psi_m \rangle \in \mathbb{R} \xrightarrow[\text{bin } T]{\text{quantization}} q[m] \in \mathbb{Z}$$

Quantization: $$q[m] = \text{sign}(a[m]) \left\lfloor \frac{|a[m]|}{T} \right\rfloor \in \mathbb{Z}$$



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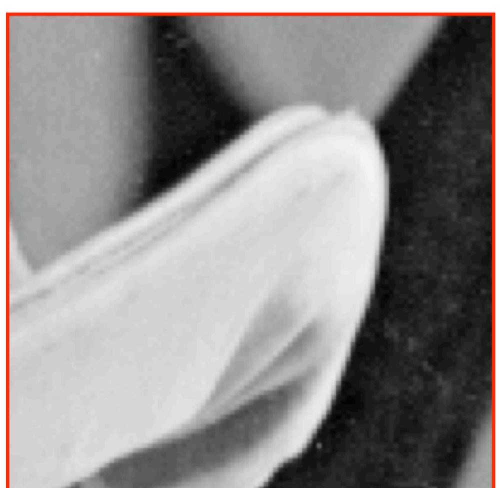

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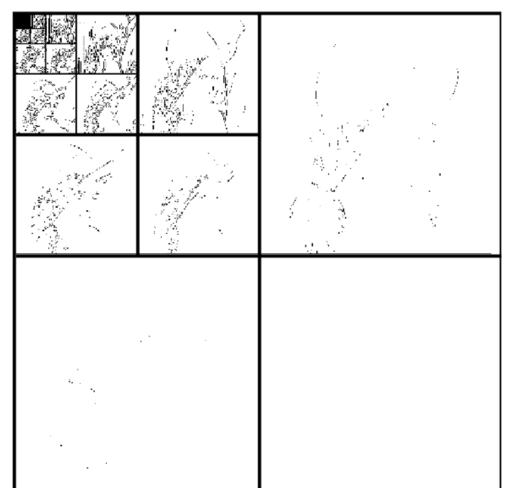

Quantized $q[m]$

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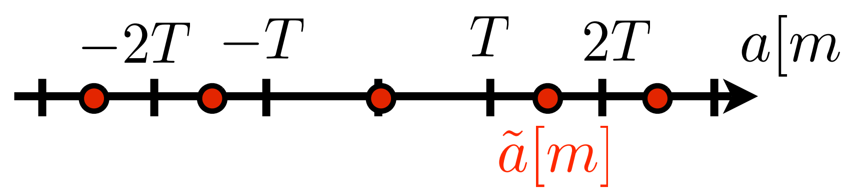

Entropic coding: use statistical redundancy (many 0's).

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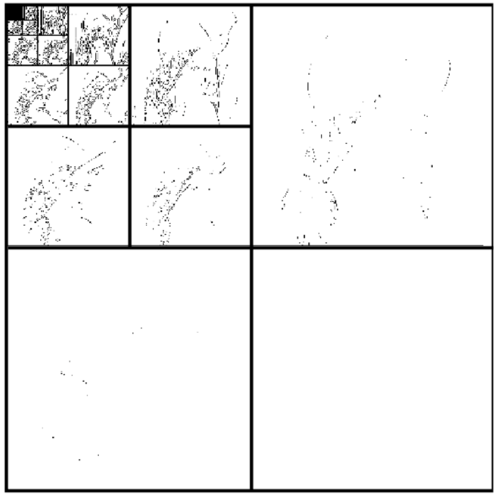

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$$\underbrace{\{\, 0,1,0,0,1,\ldots \}}_{R \text{ bits}} \xrightarrow{\text{decoding}} q[m] \in \mathbb{Z} \xrightarrow{\text{dequantization}} \tilde{a}[m]$$

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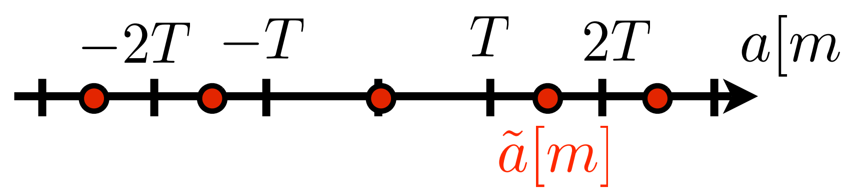
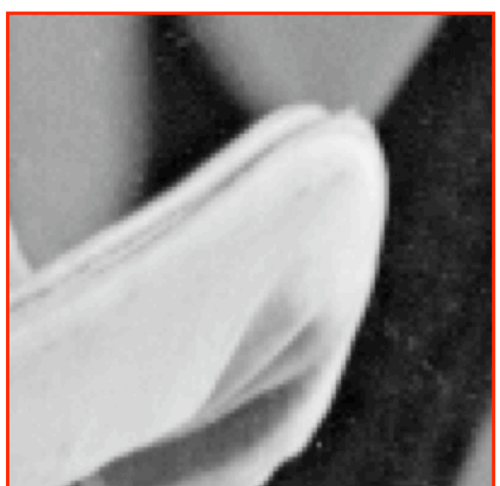


Entropic coding: use statistical redundancy (many 0's).

Dequantization: $\quad \tilde{a}[m] = \text{sign}(q[m]) \left(|q[m]| + \frac{1}{2} \right) T$

Image f



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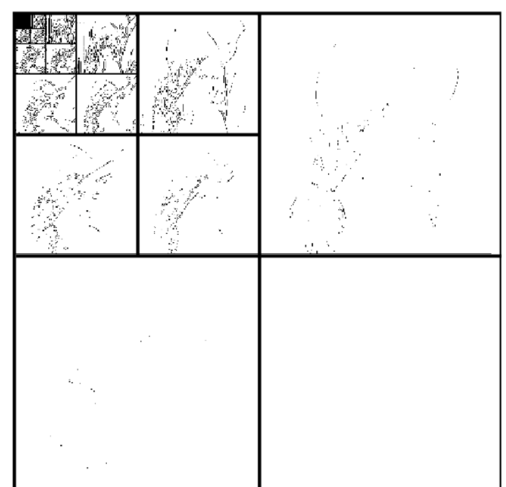

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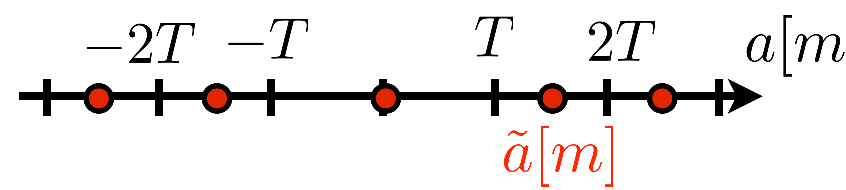
$$f_R = \sum_{m \in I_T} \tilde{a}[m]\psi_m \xleftarrow[\text{transform}]{\text{backward}} \tilde{a}[m] \xleftarrow{\text{dequantization}} q[m] \in \mathbb{Z} \xleftarrow{\text{decoding}} \{\,0,1,0,0,1,\dots\}$$

Quantization: $\quad q[m] = \mathrm{sign}(a[m]) \left\lfloor \frac{|a[m]|}{T} \right\rfloor \in \mathbb{Z}$



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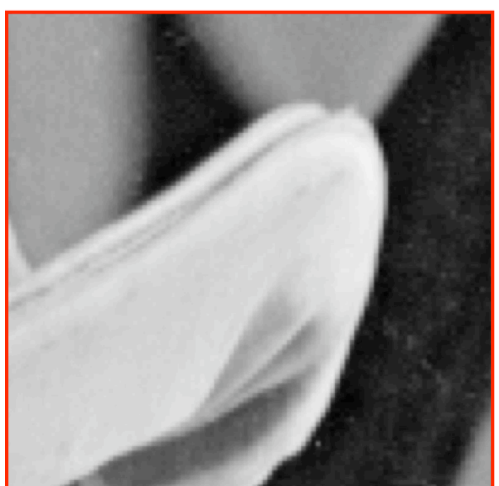

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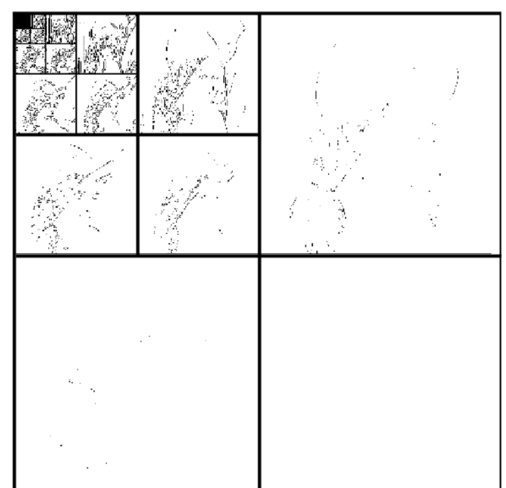
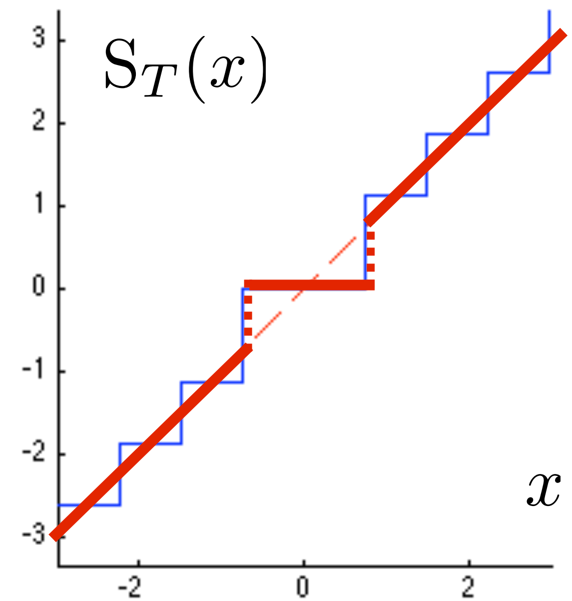


Quantized $q[m]$

f_R, R =0.2 bit/pixel

Thresholding vs. Quantizing

Thresholding: $S_T(x) = \begin{cases} x & \text{if} \quad |x| > T \\ 0 & \text{if} \quad |x| \leqslant T \end{cases}$

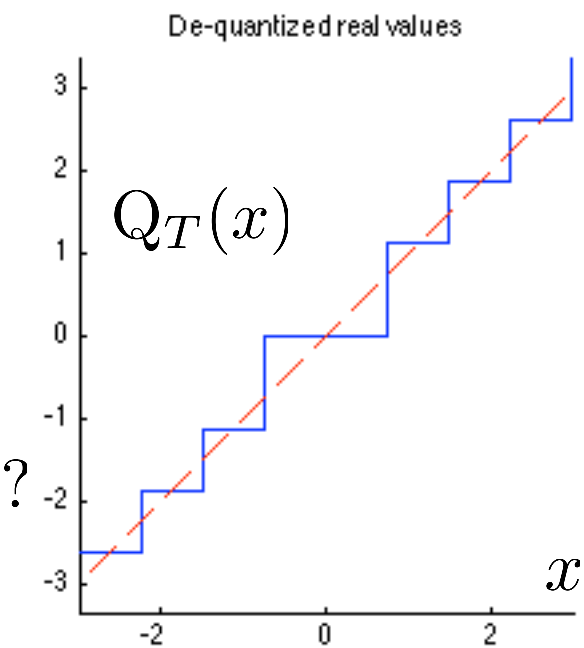


Quantization/dequantization operator:

$$Q_T(x) = \text{sign}(x)\left(\left\lfloor \frac{|x|}{T} \right\rfloor + \frac{1}{2}\right) T$$

Approximation: $f_M = \sum_m S_T(\langle f,\, \psi_m \rangle)\psi_m$

Compression: $f_R = \sum_m Q_T(\langle f,\, \psi_m \rangle)\psi_m$



Necessarily $\|f - f_M\| \leqslant \|f - f_R\|$, but how much ?

Non-linear Approximation and Compression

Performance: decompression error $\|f - f_R\|$ function of R.

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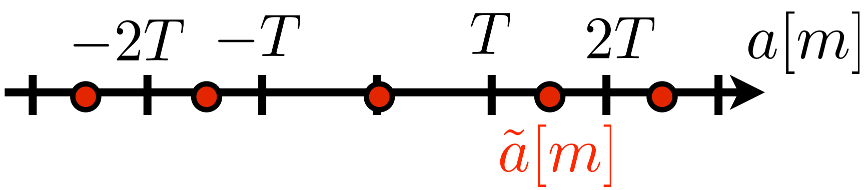

Quantization: $$q[m] = \mathrm{sign}(a[m]) \left\lfloor \frac{|a[m]|}{T} \right\rfloor \in \mathbb{Z}$$

Dequantization: $$\tilde{a}[m] = \mathrm{sign}(q[m]) \left(|q[m]| + \frac{1}{2} \right) T$$

$\Longrightarrow$ If $|a[m]| > T$: $$|a[m] - \tilde{a}[m]| \leqslant \frac{T}{2}$$

Non-linear Approximation and Compression

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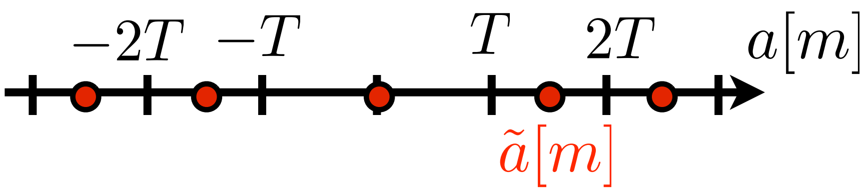

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Dequantization: $\quad \tilde{a}[m] = \mathrm{sign}(q[m]) \left(|q[m]| + \dfrac{1}{2} \right) T$

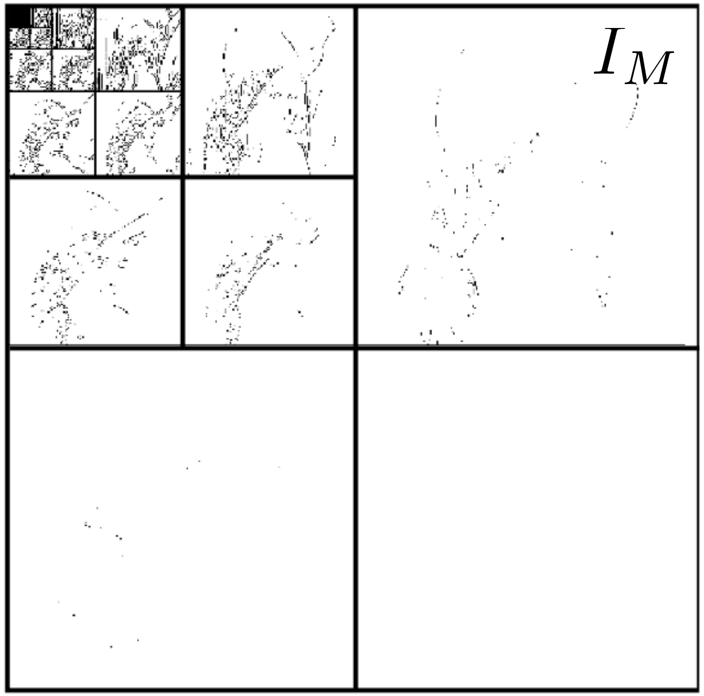
If $|a[m]| > T$: $\Longrightarrow \quad |a[m] - \tilde{a}[m]| \leqslant \dfrac{T}{2}$

$$\|f - f_R\|^2 = \sum_m (a[m] - \tilde{a}[m])^2 \leqslant \underbrace{\sum_{|a[m]|<T} |a[m]|^2}_{\|f - f_M\|^2} + \underbrace{\sum_{|a[m]|\geqslant T}}_{\# = M} \left(\frac{T}{2}\right)^2$$

Theorem: $\quad \|f - f_R\|^2 \leqslant \|f - f_M\|^2 + MT^2/4$

where $M = \# \{m \setminus \tilde{a}[m] \neq 0\}$.

A Naive Support Coding Approach



Coding: relate # bits R to # coefficients M.

Simple coding strategy:

– *Index coding:* code $I_M = \{m \setminus |\langle f, \psi_m\rangle| > T\}$

Coding a set of size $|I_M| = M$:

$$\Longrightarrow \quad R_{\text{ind}} \leqslant \log_2 \binom{N}{M} \sim M \log_2\left(\frac{N}{M}\right)$$

– *Value coding:* code $a[m] \in \{-A, \ldots, A\}$,

$$A \leqslant \frac{1}{T} \max_m |\langle f, \psi_m\rangle| \quad \Longrightarrow \quad R_{\text{val}} \leqslant C_2 M |\log_2(T)|$$

$$\Longrightarrow \quad R = R_{\text{ind}} + R_{\text{val}} \sim M(\log_2(N/M) + |\log(T)|)$$

Theorem: If $\|f - f_M\|^2 \sim M^{-\alpha}$ then $\|f - f_R\|^2 = O(R^{-\alpha} \log^{\alpha}(N/R))$.

Entropic Coders

Quantized coefficients: $q[m] \in \{-A, \ldots, A\}$ $\quad Q = 2A + 1$

$$\{q[m]\}_m \xrightarrow{\text{coding}} \{0, 1, 1, \ldots, 0, 1\} \in \{0, 1\}^R$$

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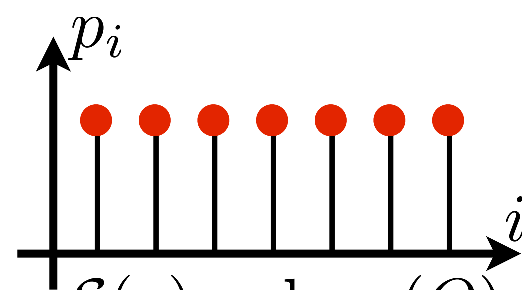
Statistical modeling: $q[m]$ i.i.d. $\mathbb{P}(q[m] = i) = p_i$.

Huffman coding: variable length code.

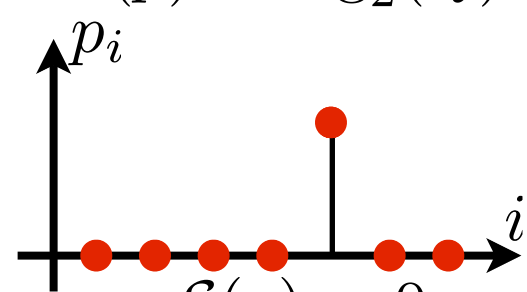
$$q[m] = i \in \{-A, \ldots, A\} \longrightarrow c_i \in \{0, 1\}^{|c_i|}$$

$$|c_i| \leqslant \lceil \log_2(p_i) \rceil \implies R \leqslant (\mathcal{E}(p) + 1)N$$

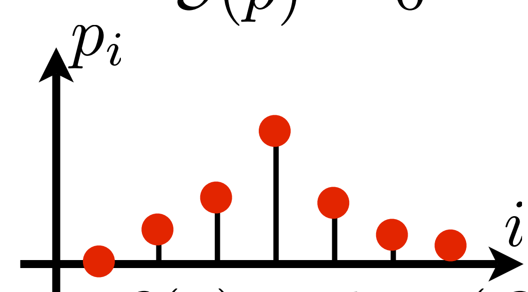
Entropy: $\quad \mathcal{E}(p) = -\sum_i p_i \log_2(p_i)$



$\mathcal{E}(p) = \log_2(Q)$



$\mathcal{E}(p) = 0$



$0 < \mathcal{E}(p) \ll \log_2(Q)$

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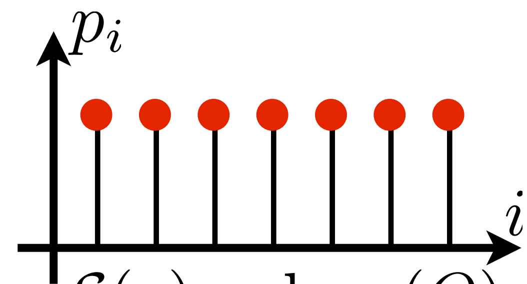

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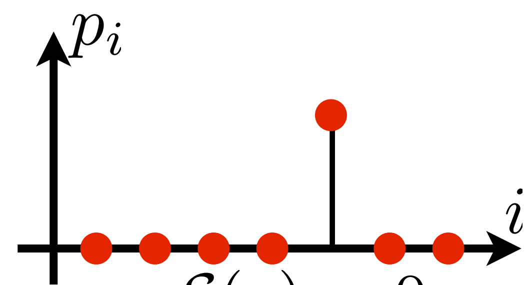

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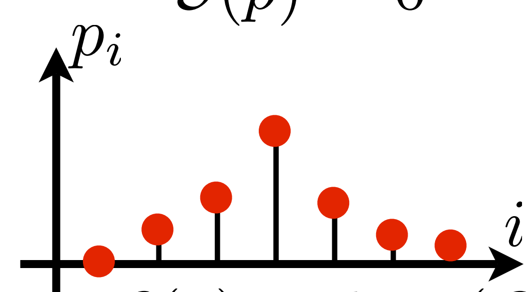


$0 < \mathcal{E}(p) \ll \log_2(Q)$

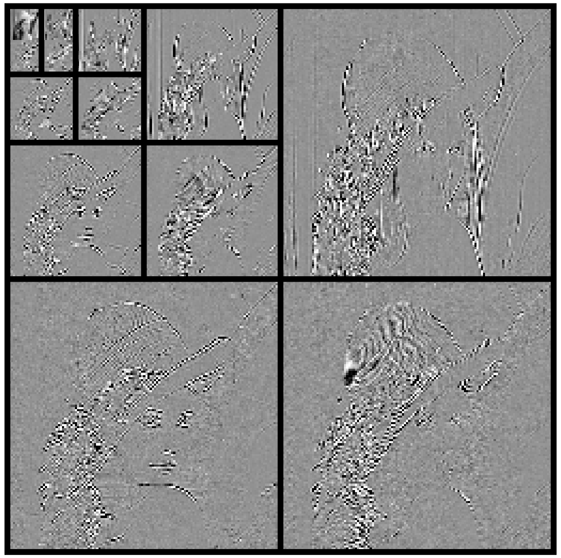
Arithmetic coding: $\quad \mathrm{R} \approx \mathcal{E}(p) N$

Fixed coder: estimate $\{p_i\}_i$ from $\{q[m]\}_m$ and send it.

Adaptive coder: estimate from the already coded data.

JPEG-2000 Overview

Wavelet transform: 7/9 biorthogonal, lifting implementation.



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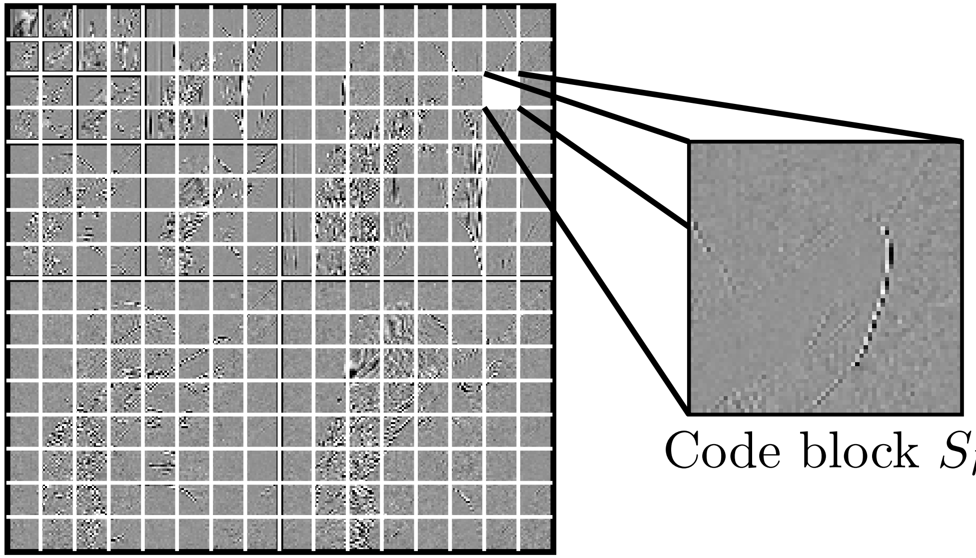
Local code block processing: enhance scalability + parallelization.



Code block S_k

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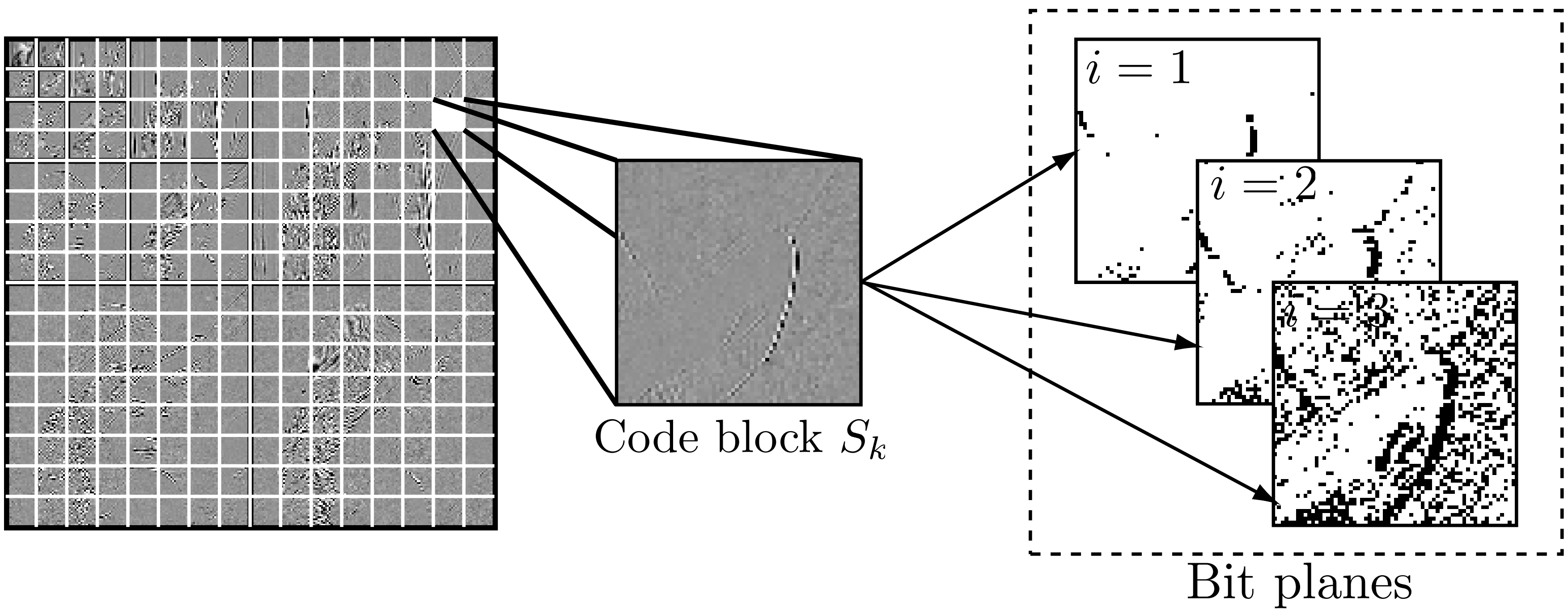
Quantization: dyadic refinement: $T_i = 2^{-i} T_0 \quad \rightarrow$ embedded coding.



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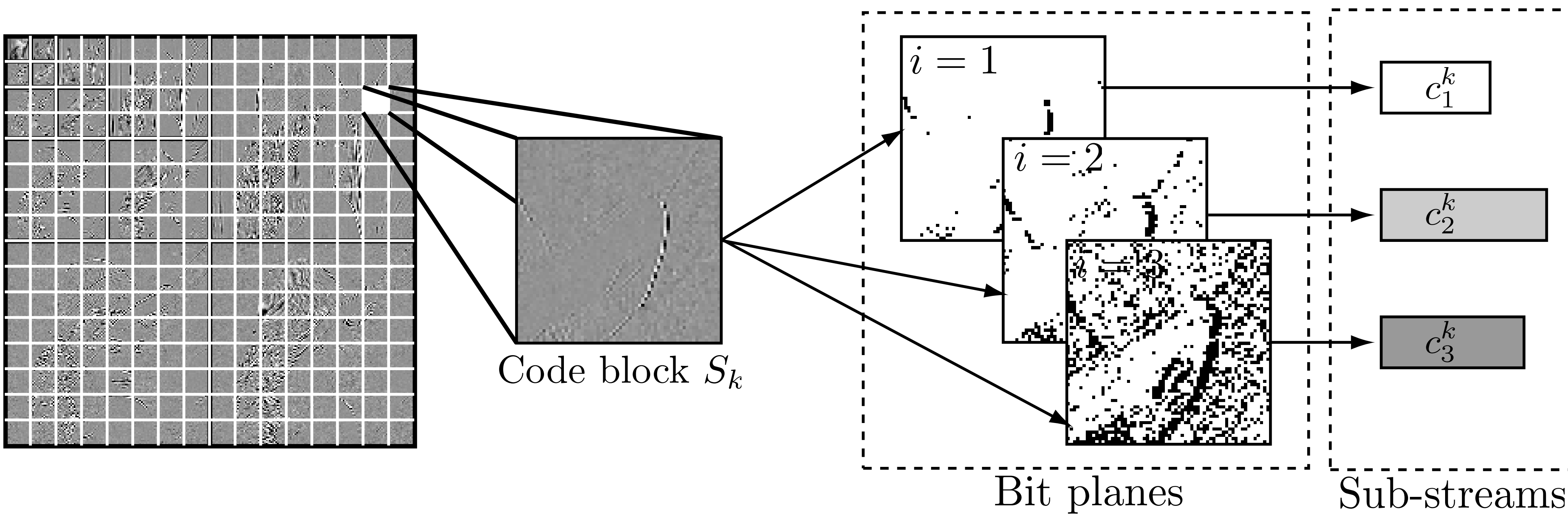
Contextual coding. $\rightarrow$ exploit spacial dependancies.



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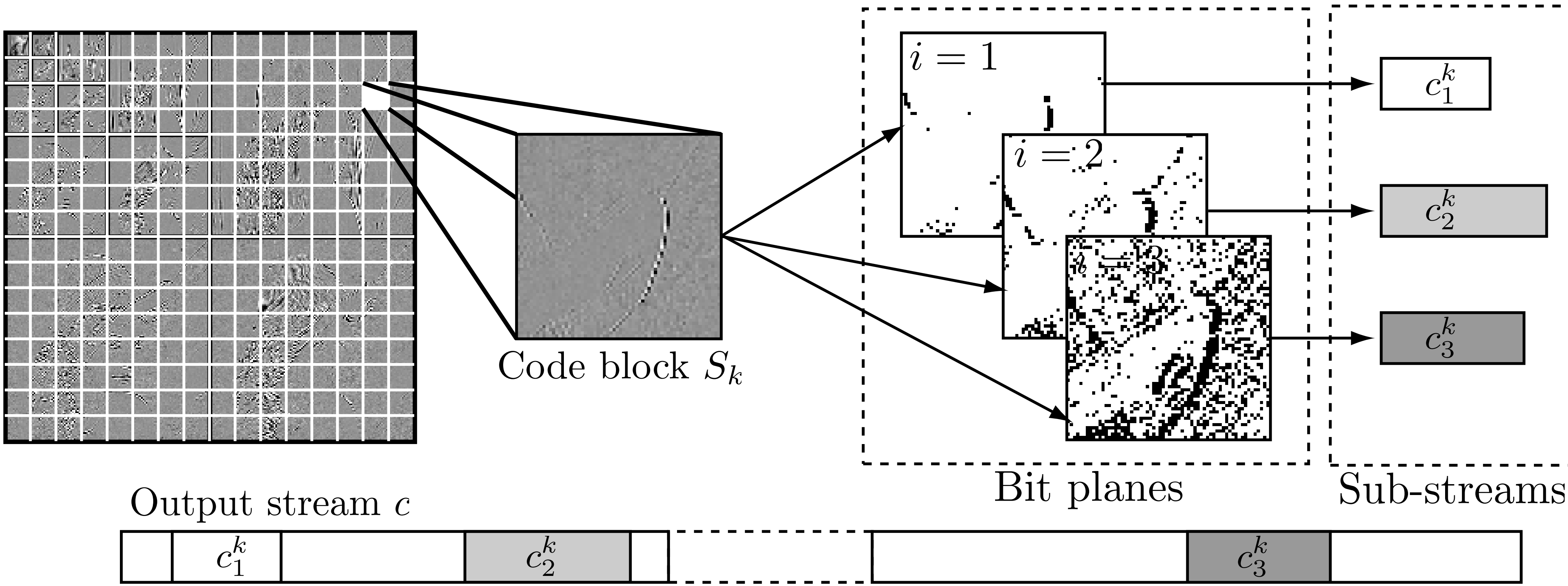
Rate-distortion management: stream packing.

$\rightarrow$ near-optimal truncation error.



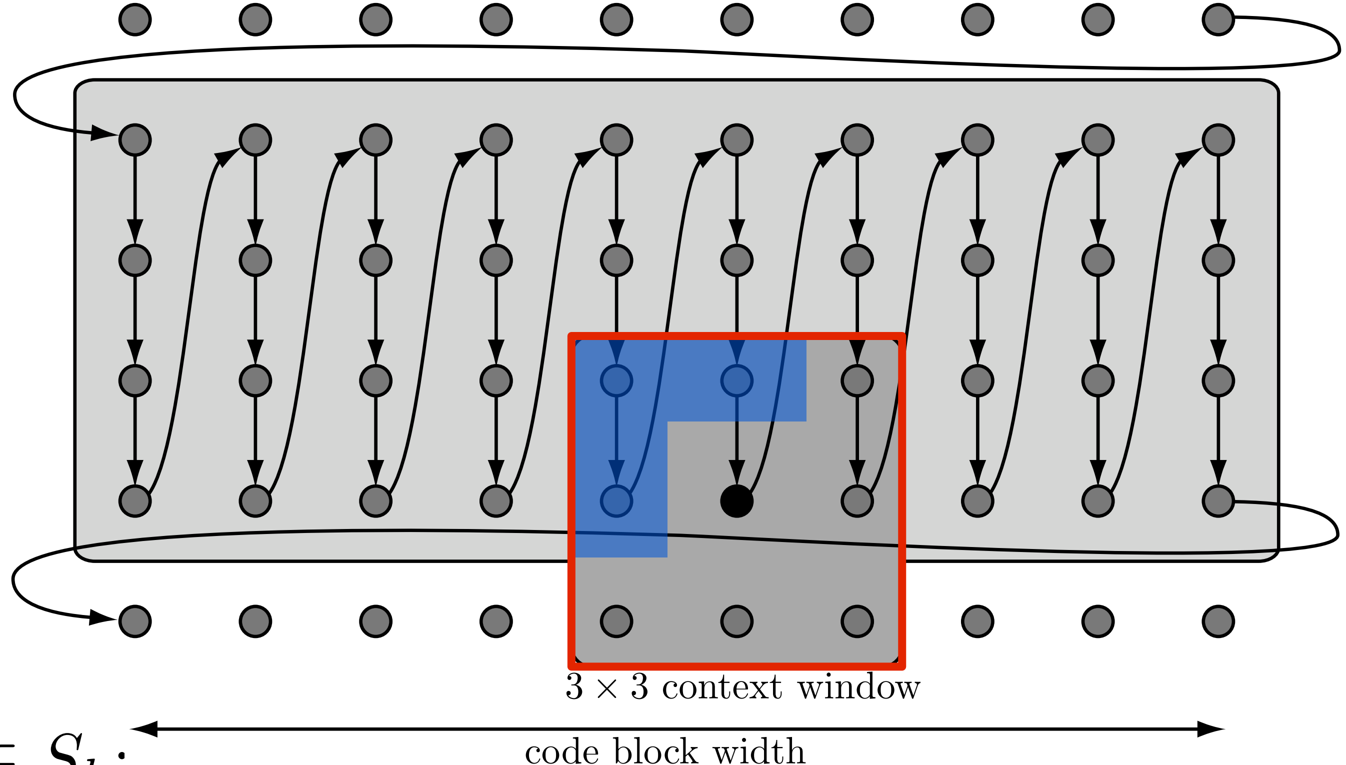
Contextual Coding

Context uses:

– previous bits at plane $i-1$.

– neighboring bits at plane $i-1$.

Sliding context window.



For each bit plane i:

For each square S_k,

For each position $m \in S_k$:

– *Signifiance coding*

If $d_j^\omega[m]$ becomes significant:

– *Sign coding*

If $d_j^\omega[m]$ was already significant:

– *Amplitude refinement.*

JPEG-2000 vs. JPEG, 0.2bit/pixel

JPEG

JPEG-2000

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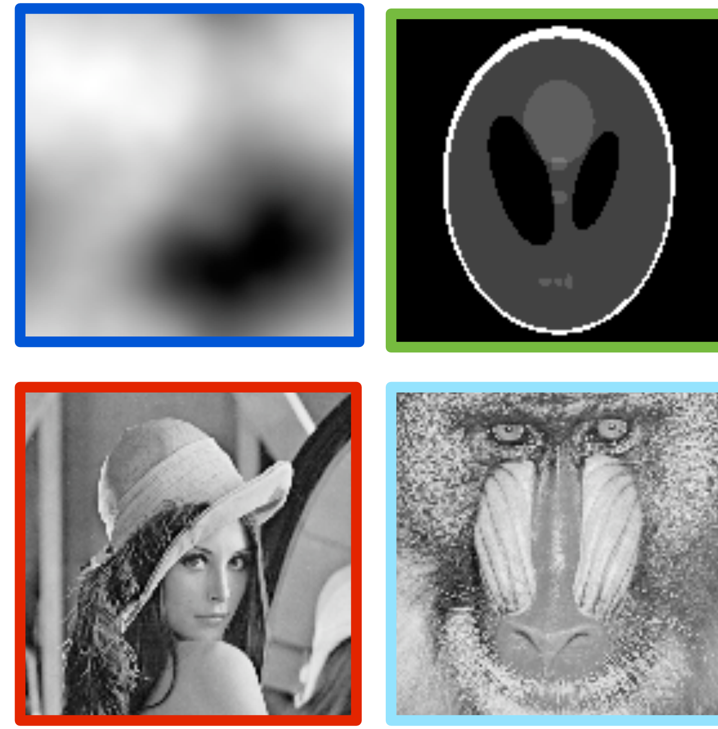
Approximation Speed

Approximation error decay:

$$\|f - f_M\|^2 \leqslant C_f M^{-\alpha}$$

(usually polynomial)



- α depend on a signal class (regular, piecewise smooth, etc).
- C_f depends on f

(norm of the signal within its class)

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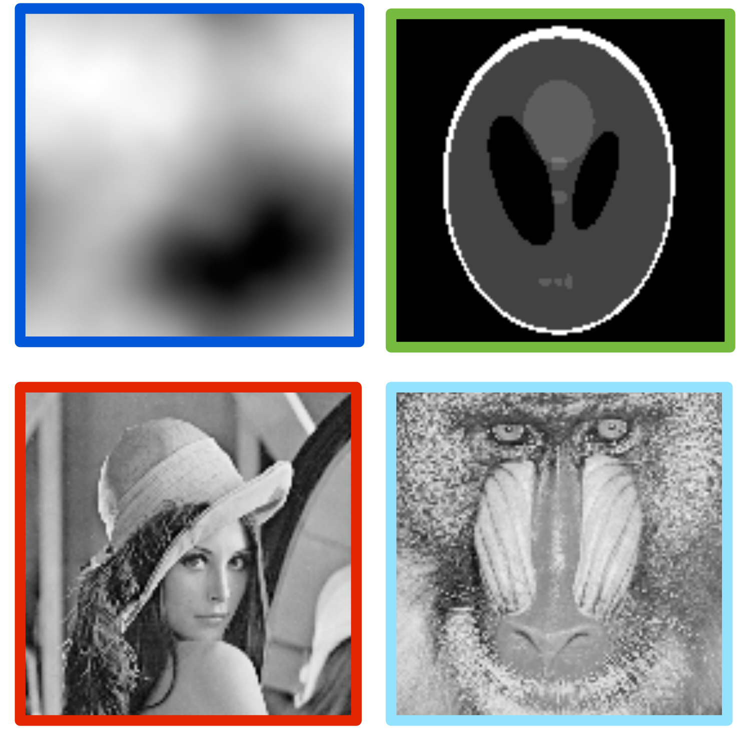
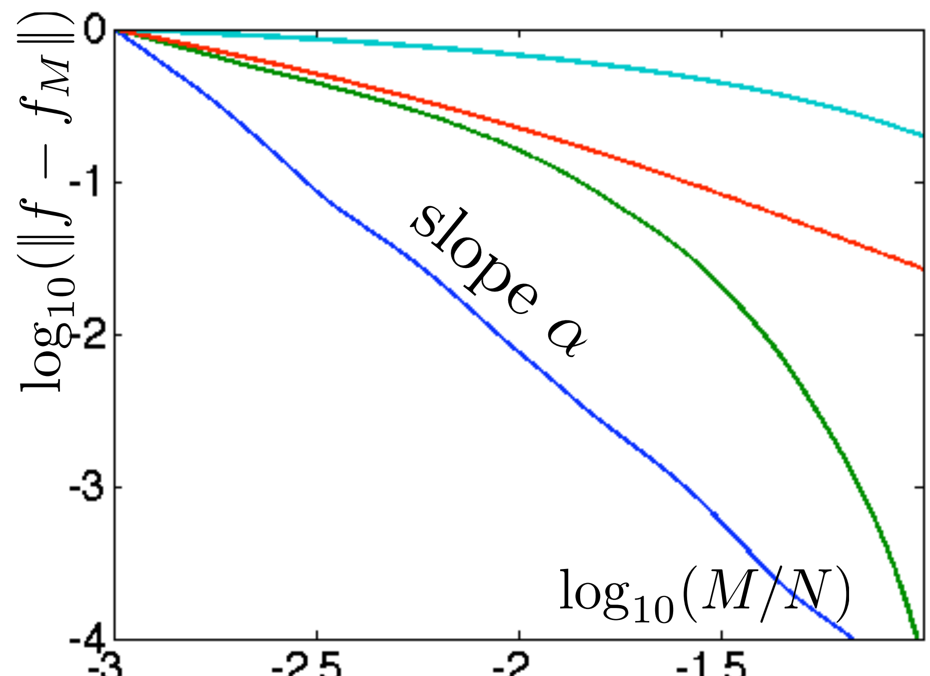
Log/Log plot: approx. affine curve

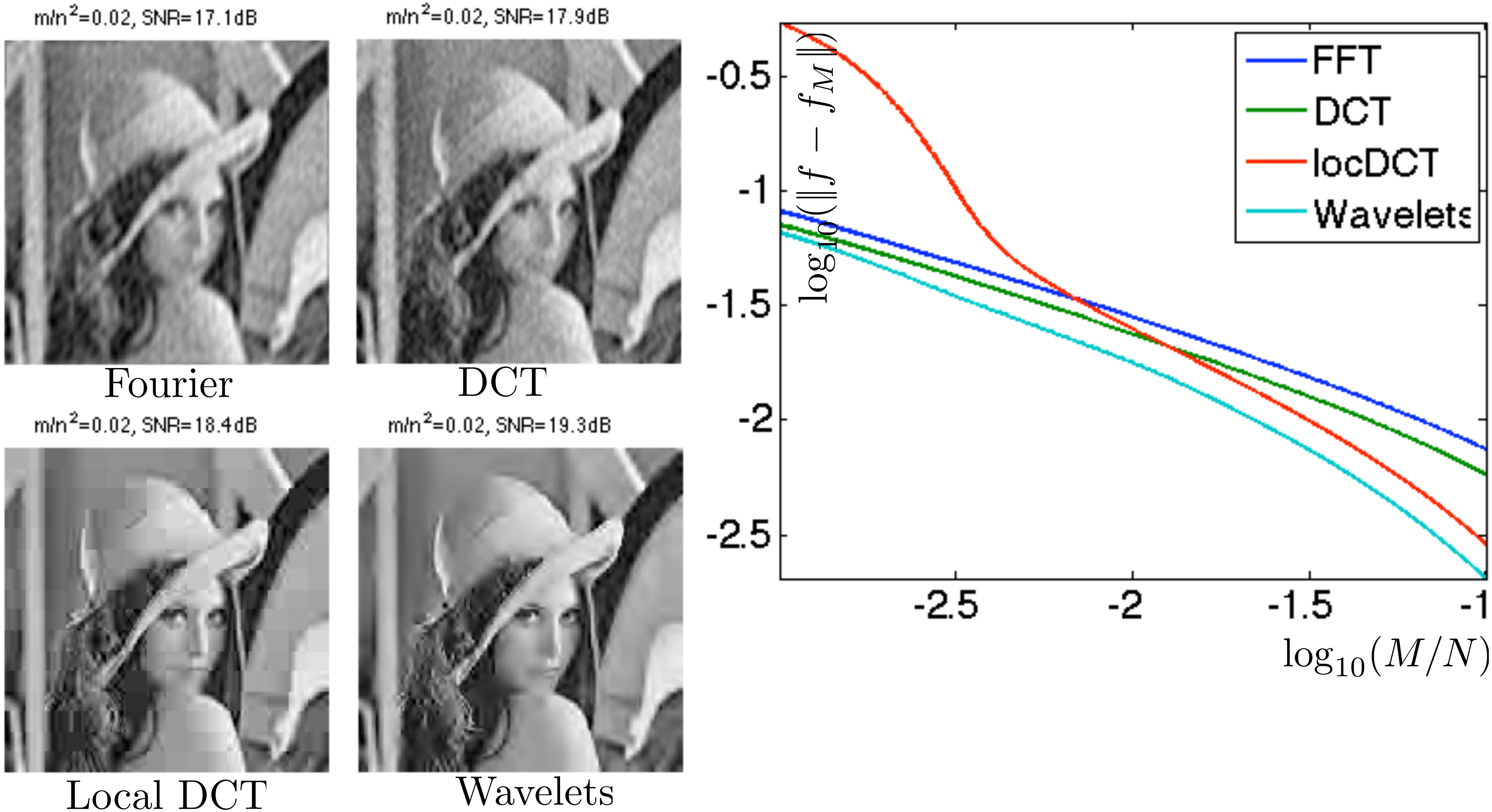
$$\log(\|f - f_M\|^2) = \text{cst} - \alpha \log(M)$$

Exemple: wavelet approximation.





Efficiency of Transforms



Build the best basis $\mathcal{B}$ to have the fastest error decay $M^{-\alpha}$

for any $f \in \Theta$ (e.g. regular signals, piecewise regular images).

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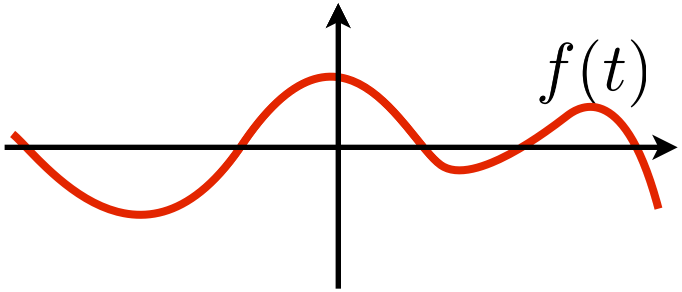
- Curvelets and Finite Elements for Cartoons

1D Fourier Approximation



1D signal : $f \in \mathrm{L}^2([0,1])$, periodic.

Fourier ortho-basis: $e_m(t) = e^{2i\pi m t}$, for $n \in \mathbb{Z}$.

Fourier coefficient: $\langle f,\, e_m \rangle = \int_0^1 f(t) e^{-2i\pi m t} \mathrm{d}t$

Linear Fourier approximation: $f_M^{\mathrm{lin}} = \sum_{m=-M/2}^{M/2} \langle f,\, e_m \rangle e_m$

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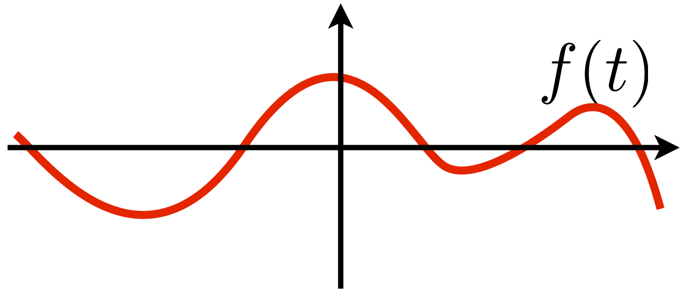

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Uniformly regular signal: $f \in \mathrm{C}^\alpha$ (α continuous derivatives).

$$\Longrightarrow \quad |\langle f,\, e_m \rangle| \leqslant |2\pi m|^{-\alpha} \|f^{(\alpha)}\|$$

$\Longrightarrow \quad \|f - f_M^{\text{lin}}\|^2$ decays fast.

Fast decay of coefficients $\Longrightarrow$ good linear approximation.

Sobolev and Fourier

Uniformly C^α signal $\quad\Longrightarrow\quad f^{(\alpha)} \in L^2([0,1])$.

Sobolev α-regular signal: $\|f^{(\alpha)}\| < +\infty$.

Sobolev norm: $\quad \|f^{(\alpha)}\|^2 = \sum_m |2\pi m|^{2\alpha} |\langle f,\, e_m\rangle|^2 < +\infty$

$$\|f^{(\alpha)}\|^2 = \sum_m |2\pi m|^{2\alpha} |\langle f,\, e_m\rangle|^2 \geqslant \sum_{|m|>M/2} |2\pi m|^{2\alpha} |\langle f,\, e_m\rangle|^2$$

$$\geqslant (\pi M)^{2\alpha} \sum_{m>M/2} |\langle f,\, e_m\rangle|^2 \geqslant (\pi M)^{2\alpha} \|f - f_M^{\text{lin}}\|^2$$

Theorem: if f is Sobolev α-regular,

$$\|f - f_M^{\text{lin}}\|^2 \leqslant C \|f^{(\alpha)}\|^2 M^{-2\alpha}$$

Linear approximation $\approx$ non-linear approximation.

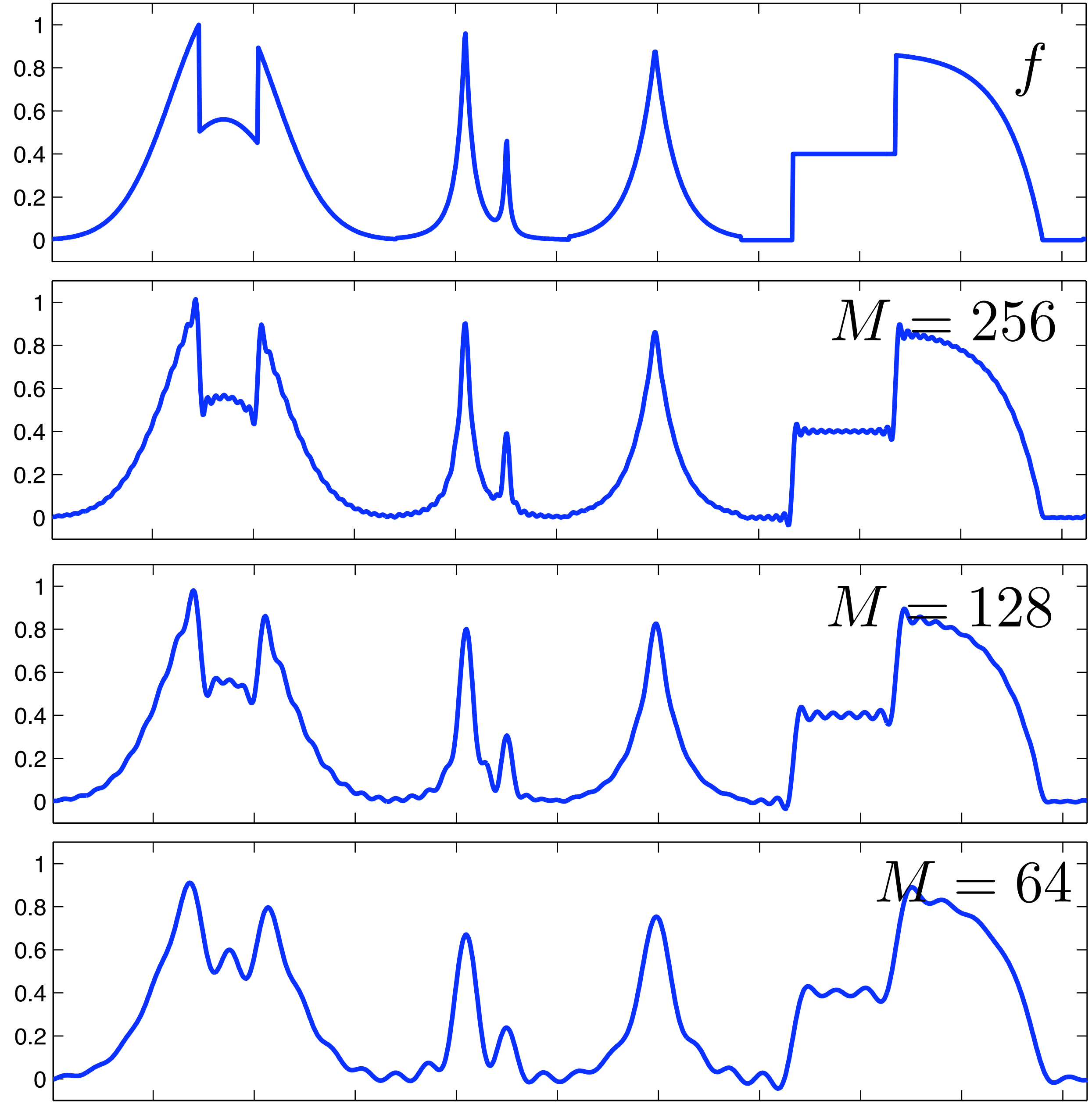
Singularities and Fourier

Piecewise regular signal

$$\|f - f_M\|^2 \leqslant C_f M^{-1}$$

Linear approximation

$\approx$ non-linear approximation.



Sobolev for Images

2D Sobolev norm:

$$\|f^{(\alpha)}\|^2 = \sum_{m \in \mathbb{Z}^2} \|2m\pi\|^{2\alpha} |\langle f,\, e_m \rangle|^2$$

Linear/non-linear approximation: $\|f - f_M\|^2 \leqslant C \|f^\alpha\|^2 M^{-\alpha}$

Sobolev for Images

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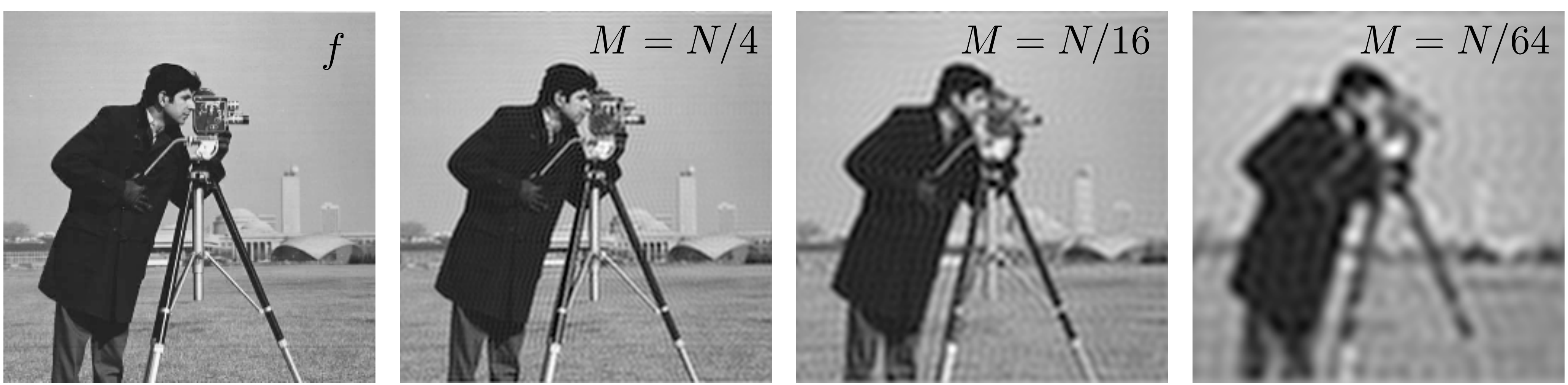
Piecewise regular image: $\|f - f_M\|^2 \leqslant C_f M^{-1/2}$

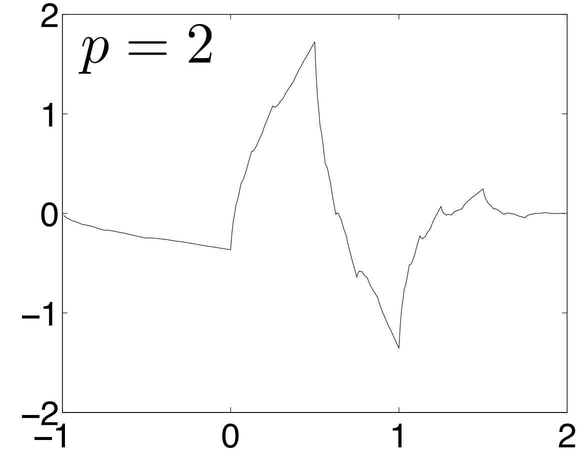


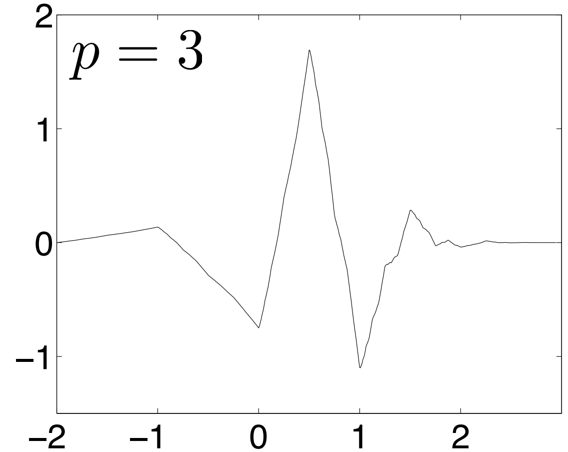
Overview

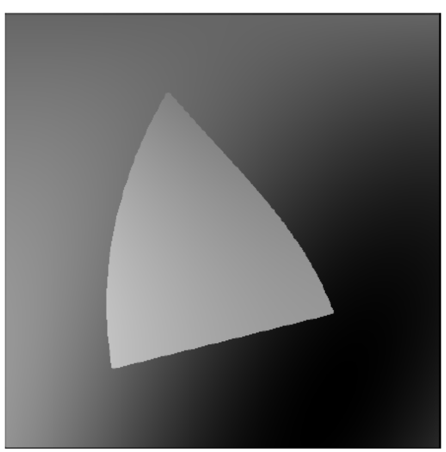
- Approximation and Compression
- Decay of Approximation Error
- Fourier for Smooth Functions
- **Wavelet for Piecewise Smooth Functions**
- Curvelets and Finite Elements for Cartoons

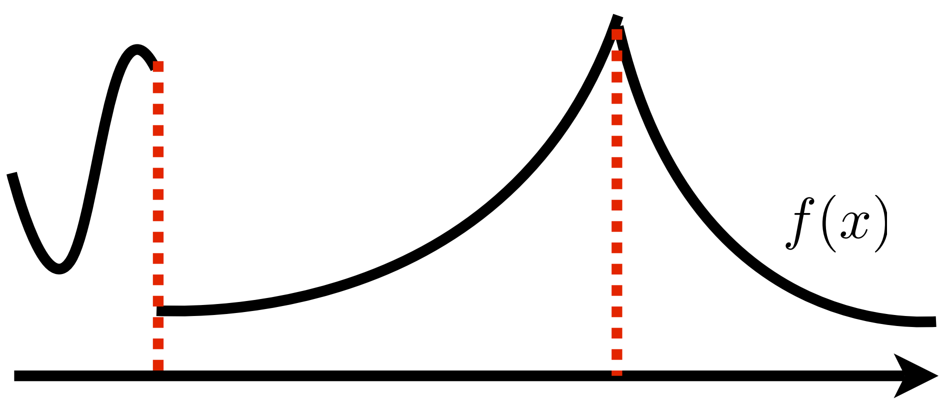
Magnitude of Wavelet Coefficients

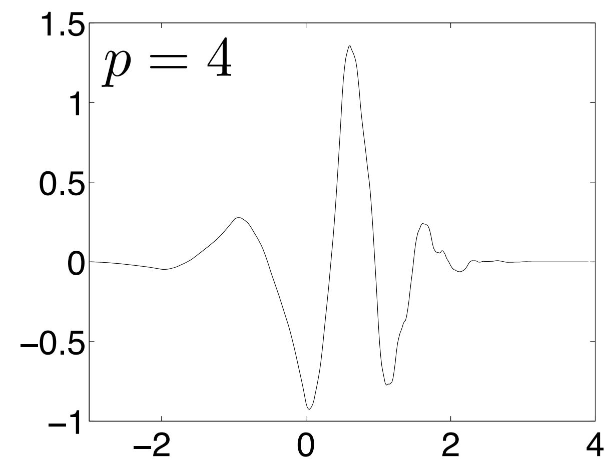
Vanishing moments: $\forall\, k < p, \ \int \psi(x) x^k \mathrm{d}x = 0$











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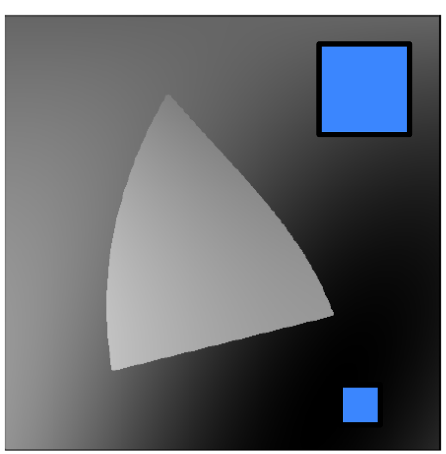
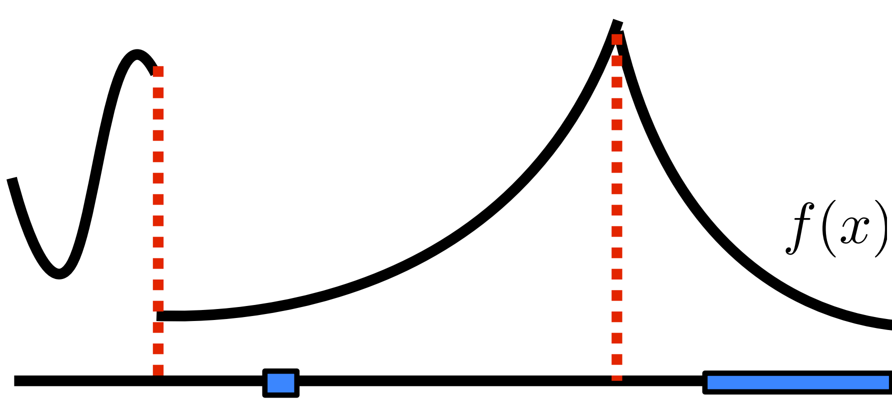
$$\langle f,\, \psi_{j,n} \rangle = \frac{1}{2^{j\frac{d}{2}}} \int f(x) \psi\left(\frac{x - 2^j n}{2^j} \right) \mathrm{d}x = 2^{j\frac{d}{2}} \int R(2^j t) \psi(t) \mathrm{d}t$$

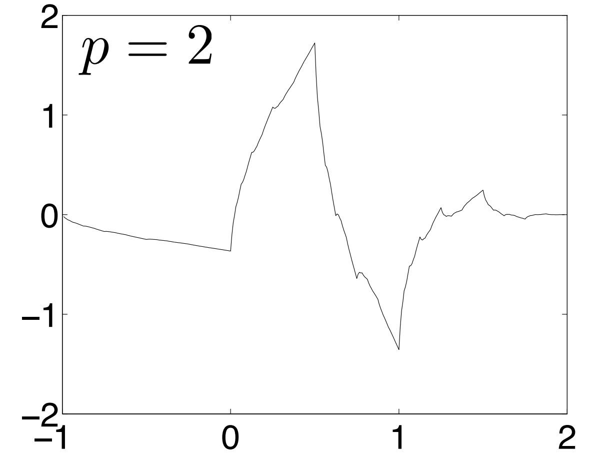
$$f(x) = P(x - 2^j n) + R(x - 2^j n) = P(2^j t) + R(2^j t)$$

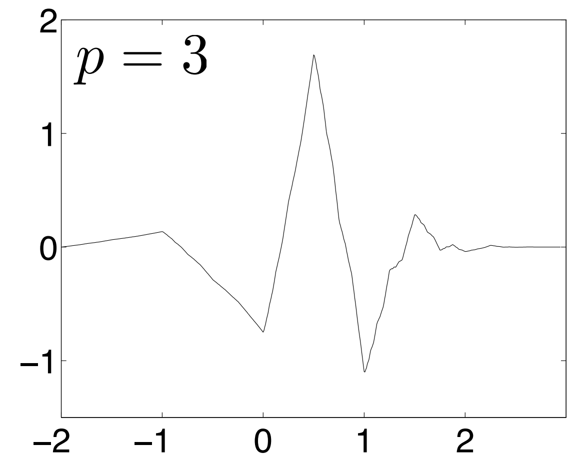
where $\quad \deg(P) < \alpha \quad$ and $\quad |R(x)| \leqslant C_f \|x\|^\alpha$

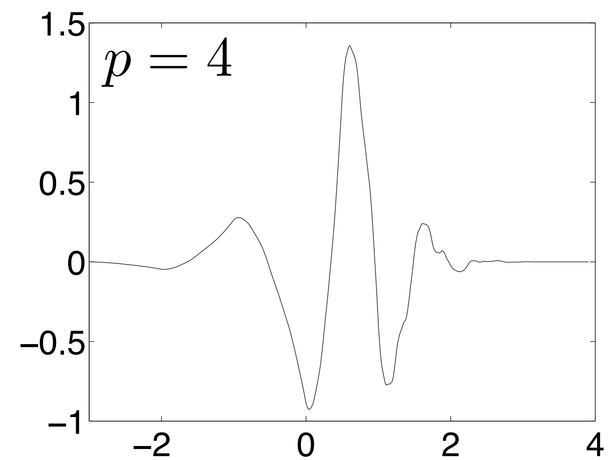
$$\boxed{|\langle f,\, \psi_{j,n} \rangle| \leqslant C_f \|\psi\|_1 2^{j(\alpha + d/2)}}$$











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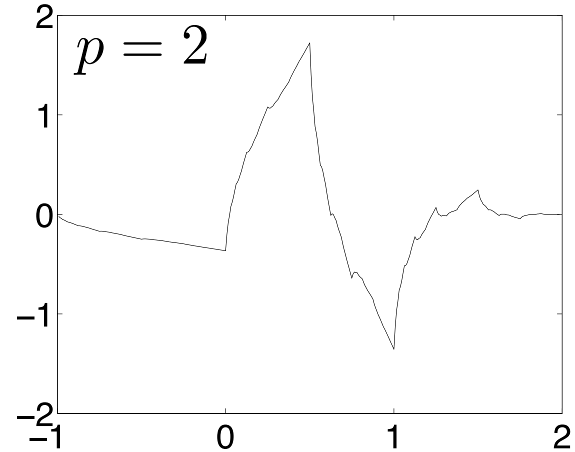
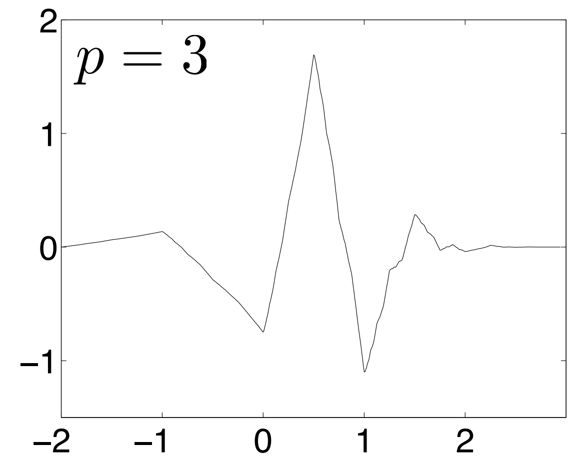
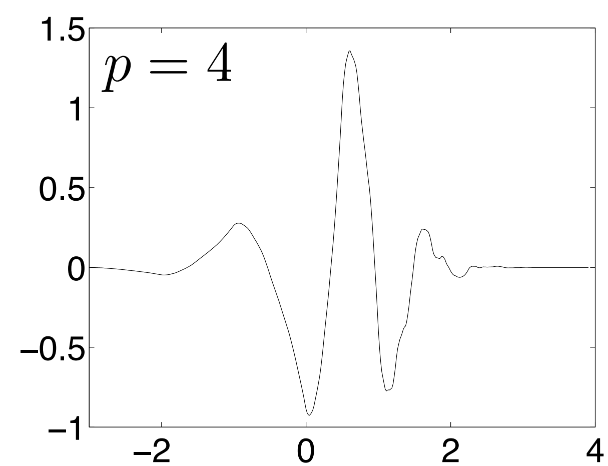
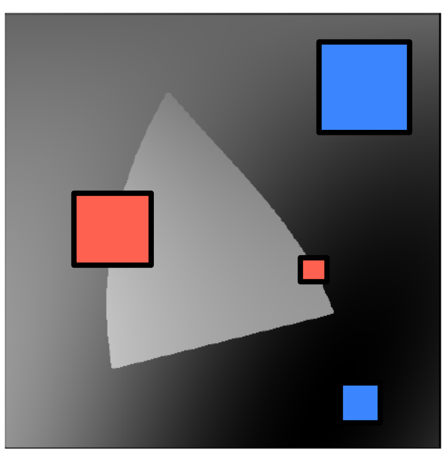
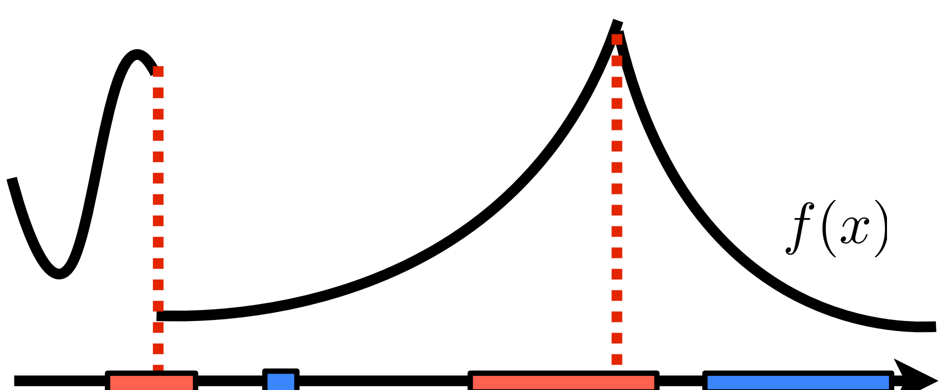




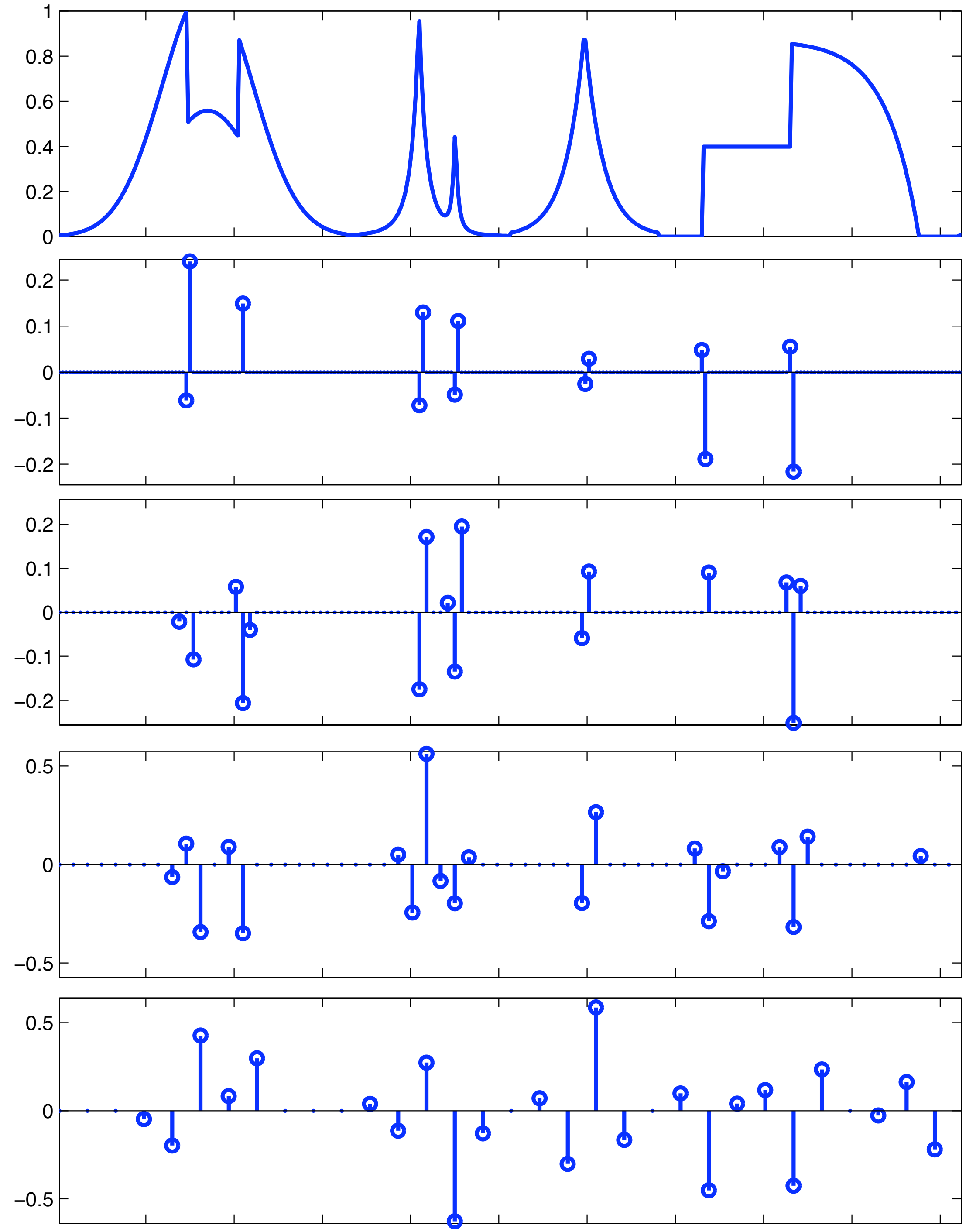






1D Wavelet Coefficient Behavior

Large wavelet coefficients are clustered around singularities.



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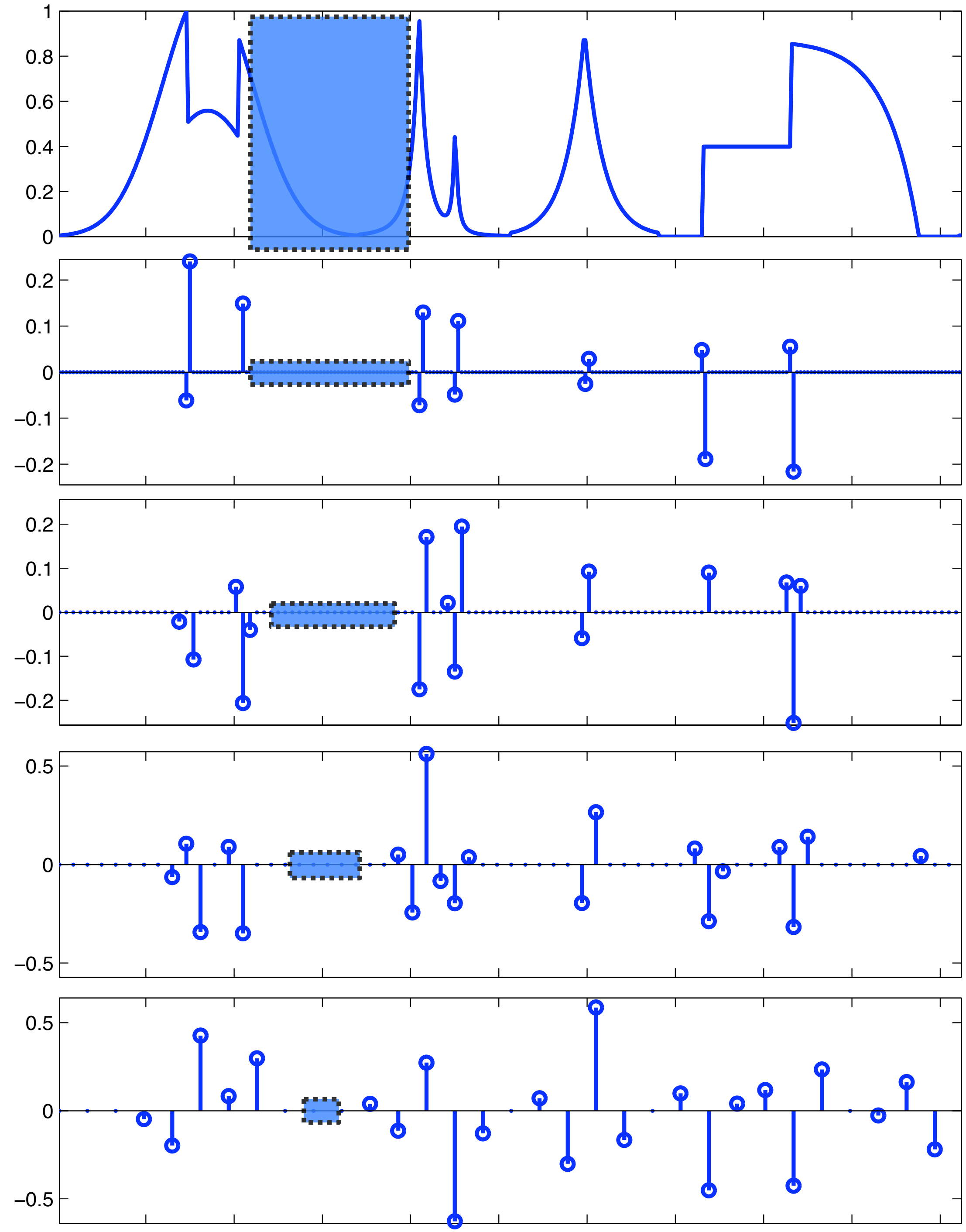
If f is C^α in $supp(\psi_{j,n})$, then

$$|\langle f,\, \psi_{j,n}\rangle| \leqslant 2^{j(\alpha+1/2)} \|f\|_{\mathrm{C}^\alpha} \|\psi\|_1$$



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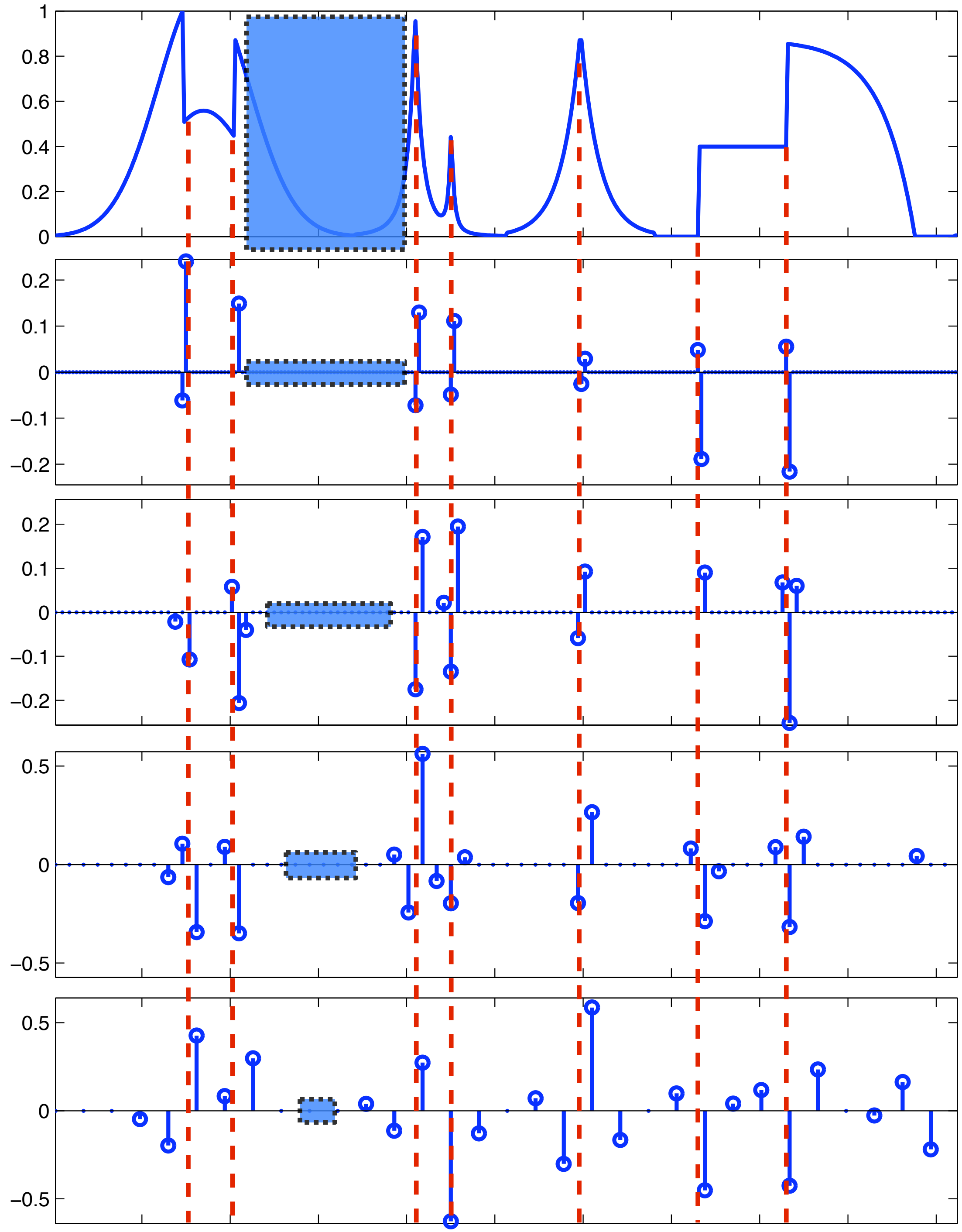
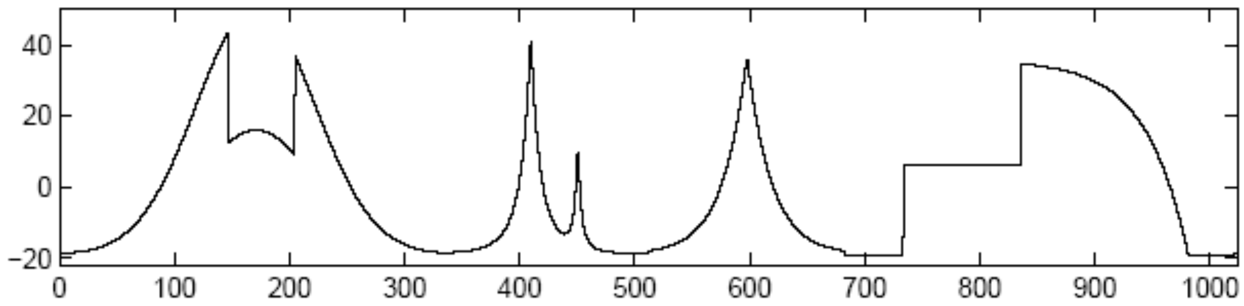
If f is bounded (e.g. around a singularity), then

$$|\langle f,\, \psi_{j,n}\rangle| \leqslant 2^{j/2} \|f\|_\infty \|\psi\|_1$$



Piecewise Regular Functions in 1D



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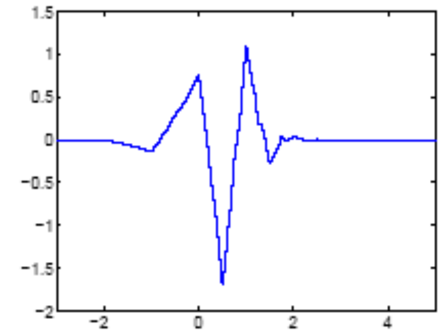


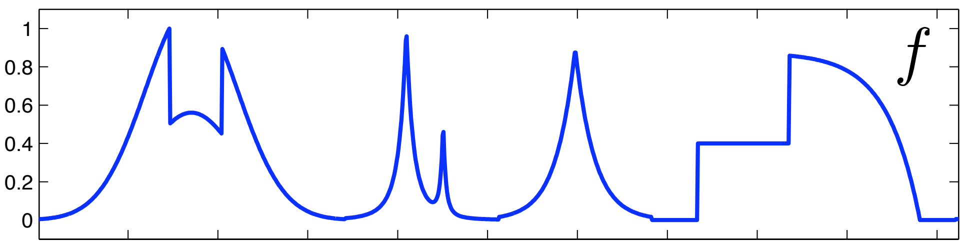
Theorem: If f is C^α outside a finite set of discontinuities:

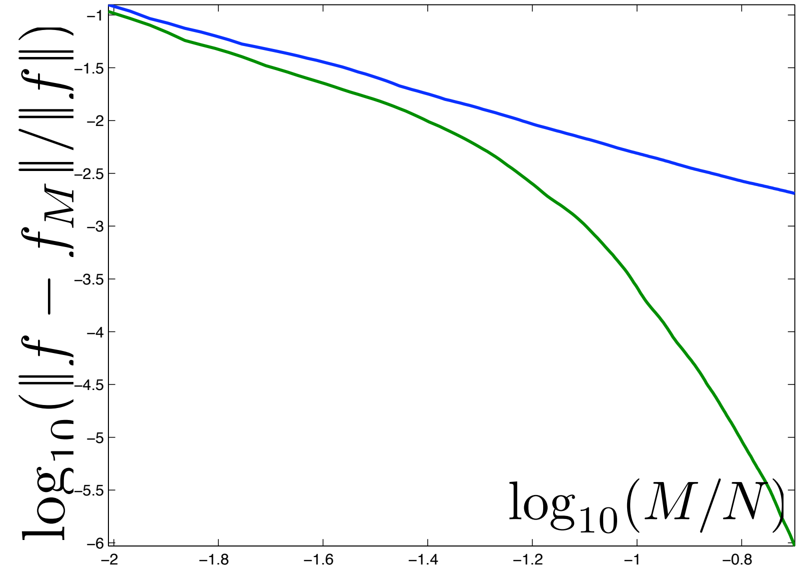
$$\varepsilon^n[M] = \|f - f_M^n\|^2 = \begin{cases} O(M^{-1}) & \text{(Fourier)}, \\ O(M^{-2\alpha}) & \text{(wavelets)}. \end{cases}$$

For Fourier, linear$\approx$non-linear, sub-optimal.

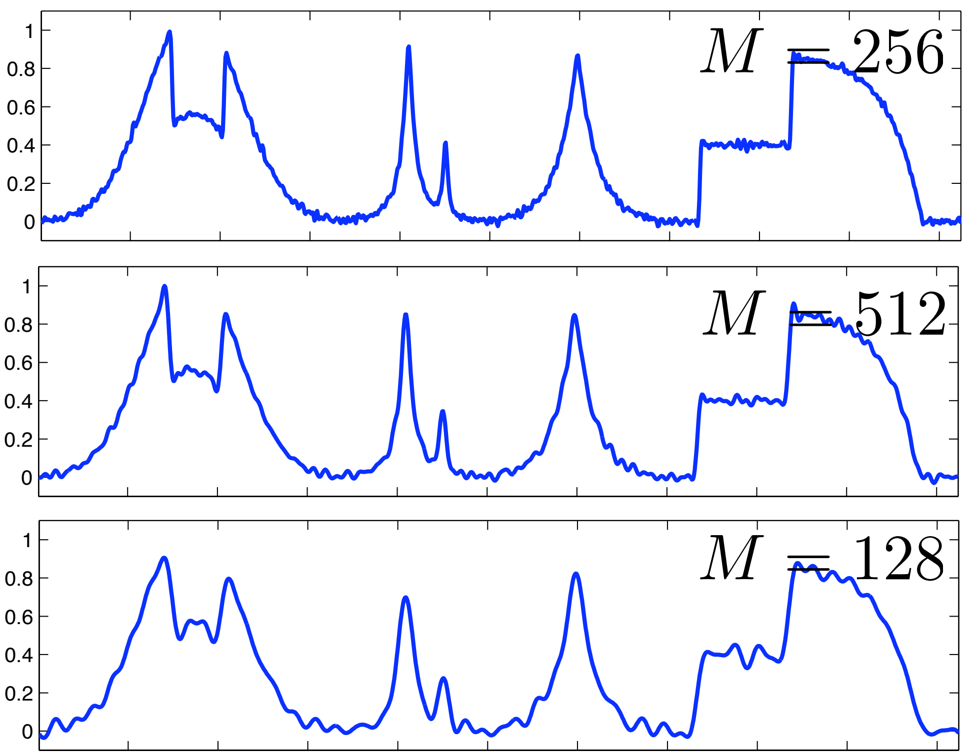
For wavelets, linear$\ll$non-linear, optimal.

Examples of 1D Approximations

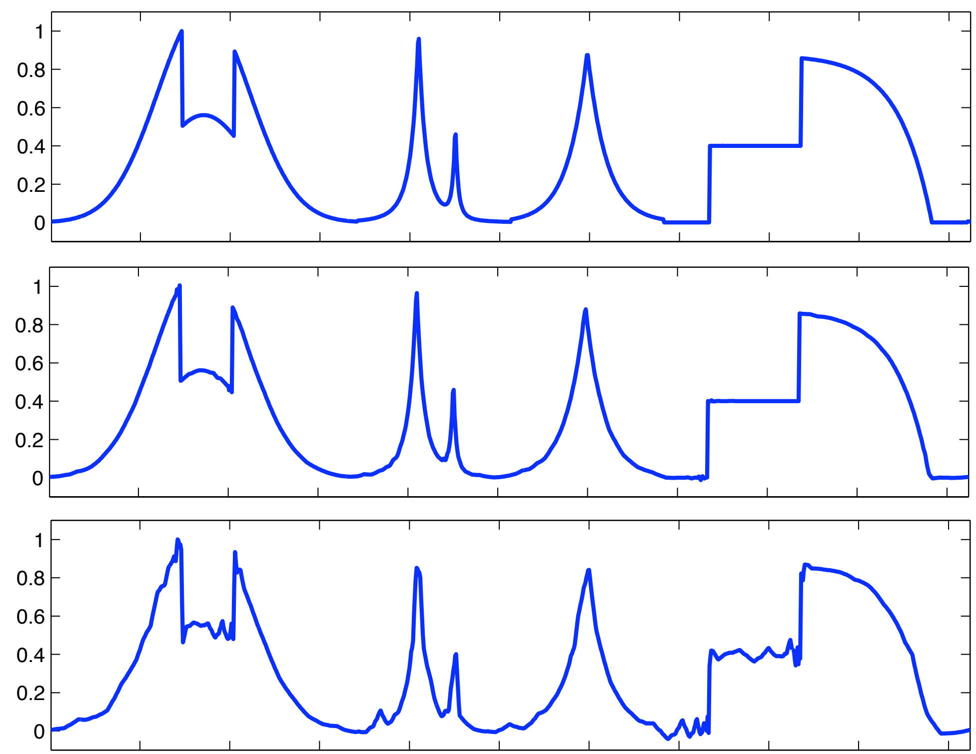




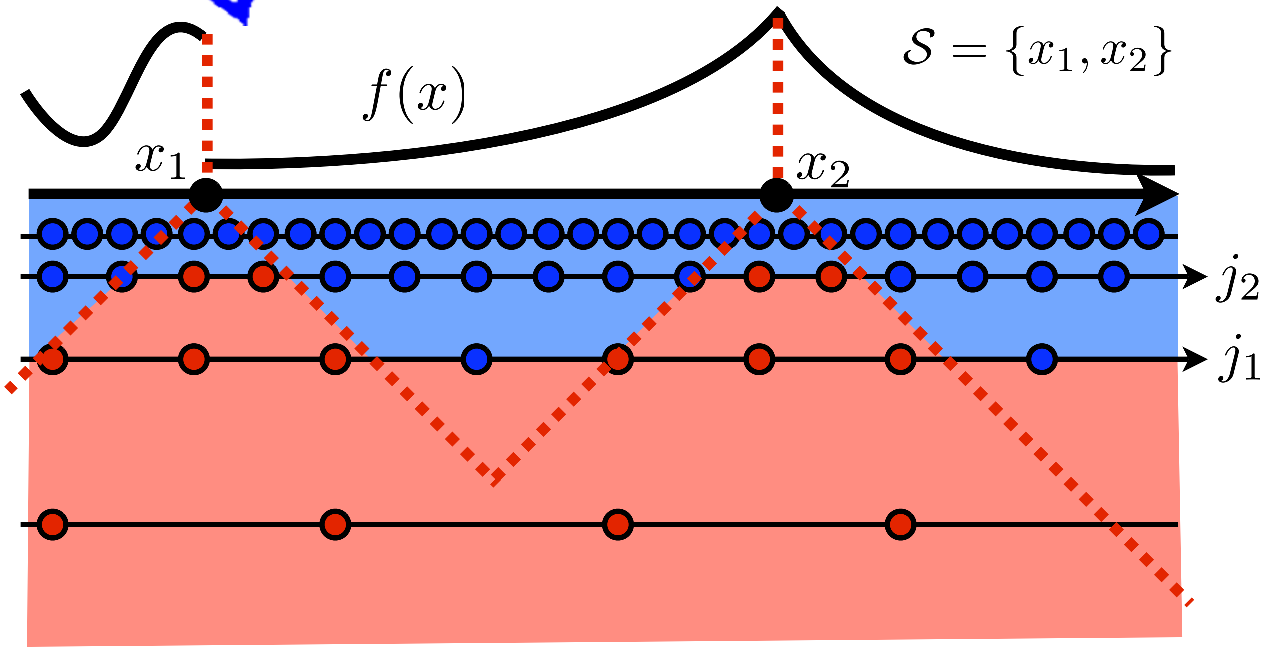
Fourier



Wavelets



Localizing the Singular Support



Small coefficient $|\langle f,\, \psi_{jn} \rangle| < T$

Large coefficient

Singular support:

$$\mathcal{C}_j = \{n \setminus \operatorname{supp}(\psi_{j,n}) \cap \mathcal{S} \neq \emptyset\}$$

$$|\mathcal{C}_j| \leqslant K|\mathcal{S}| = \text{constant}$$

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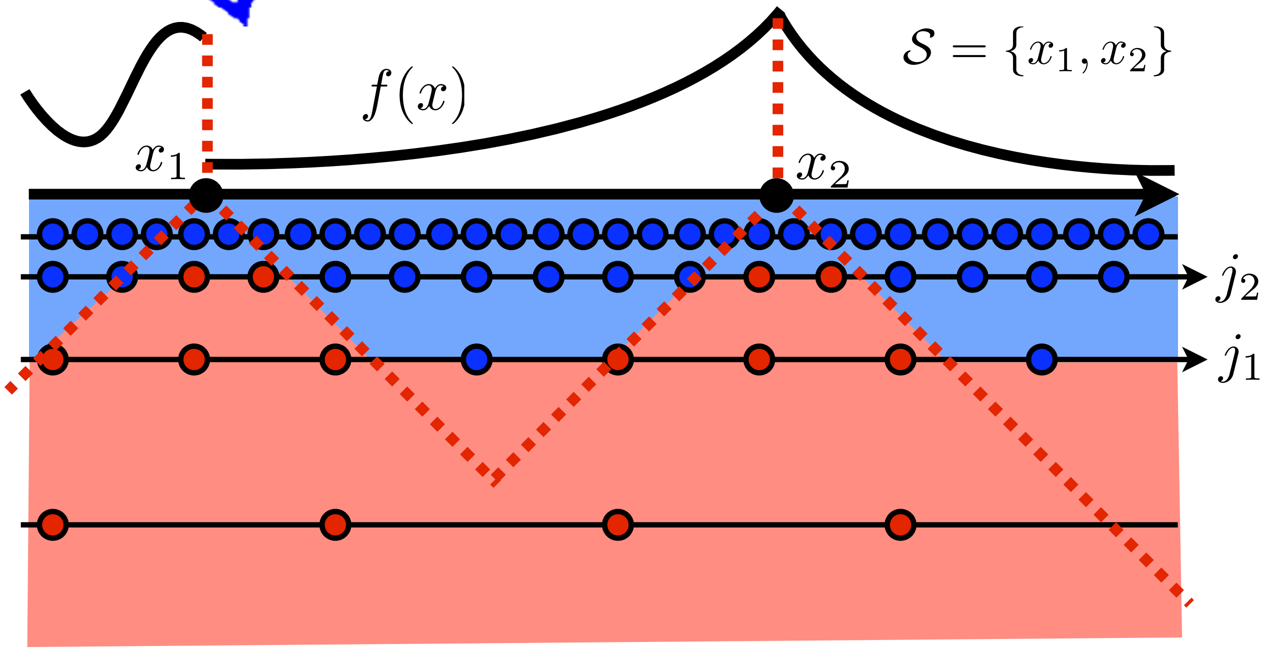

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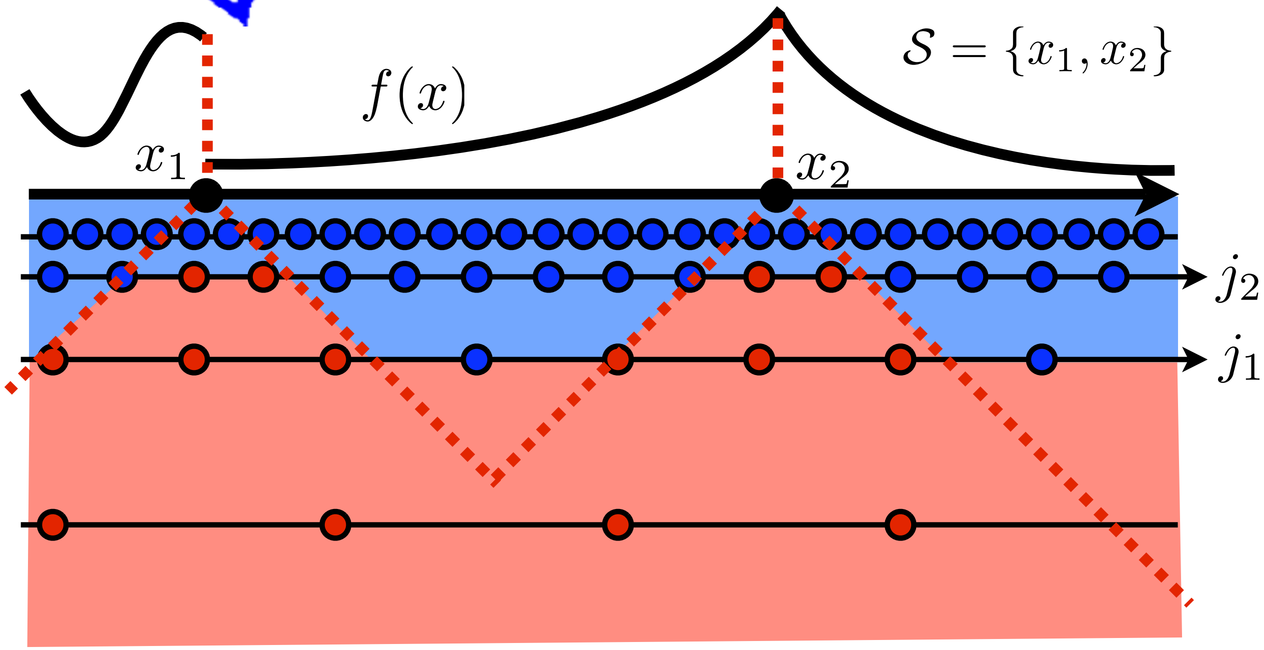
Coefficient behavior:

Regular, $n \in \mathcal{C}_j^c$: $\quad |\langle f,\, \psi_{j,n}\rangle| \leqslant C 2^{j(\alpha+1/2)}$

Singular, $n \in \mathcal{C}_j$: $\quad |\langle f,\, \psi_{j,n}\rangle| \leqslant C 2^{j/2}$

Localizing the Singular Support



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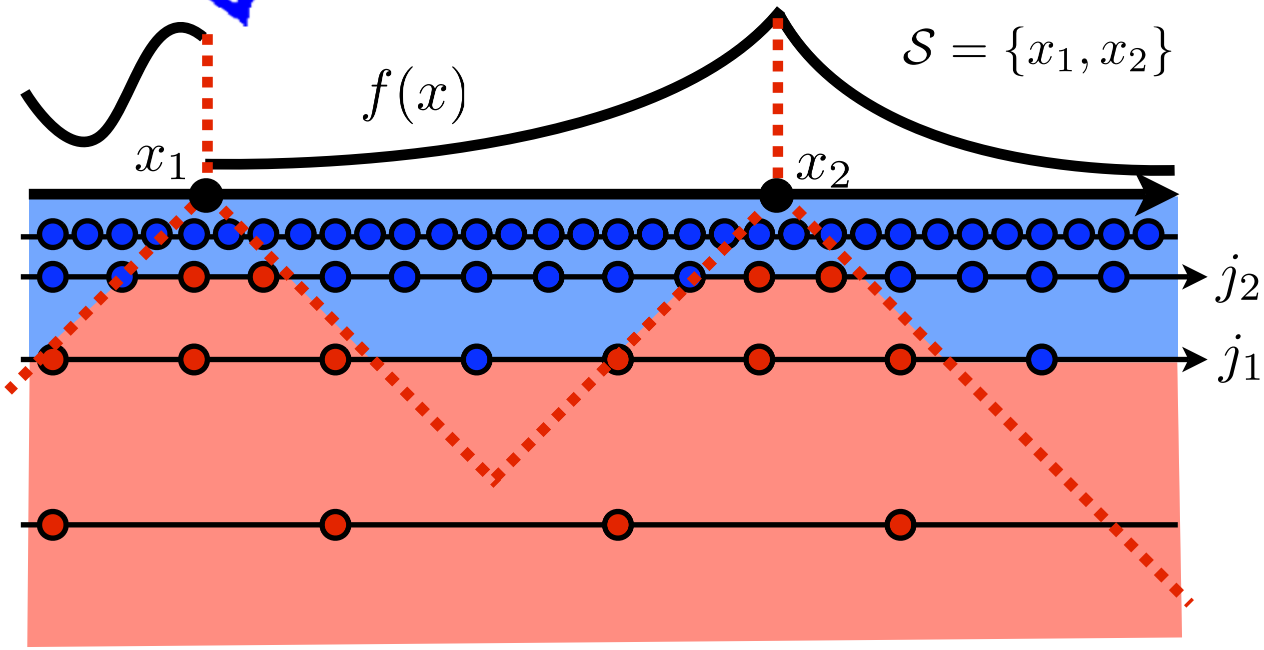
Cut-off scales (depends on T):

Singular: $\quad 2^{j_1} = (T/C)^{\frac{1}{\alpha+1/2}}$

Regular: $\quad 2^{j_2} = (T/C)^2$

Localizing the Singular Support



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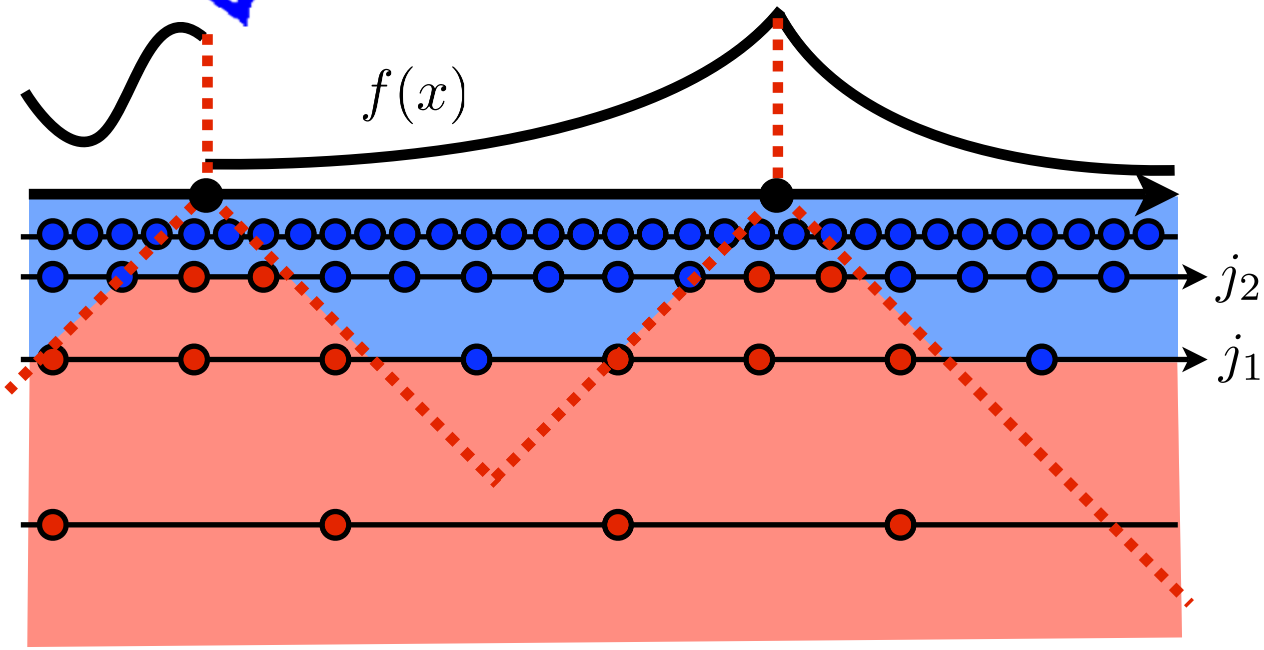
Cut-off scales (depends on T):

Singular: $\quad 2^{j_1} = (T/C)^{\frac{1}{\alpha+1/2}}$

Regular: $\quad 2^{j_2} = (T/C)^2$

Hand-made approximate: $\quad \tilde{f}_M = \sum_{j \geqslant j_2} \sum_{n \in \mathcal{C}_j} \langle f,\, \psi_{j,n}\rangle \psi_{j,n} + \sum_{j \geqslant j_1} \sum_{n \in \mathcal{C}_j^c} \langle f,\, \psi_{j,n}\rangle \psi_{j,n}$

Computing Error and #Coefficients



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$$2^{j_2} = (T/C)^2$$

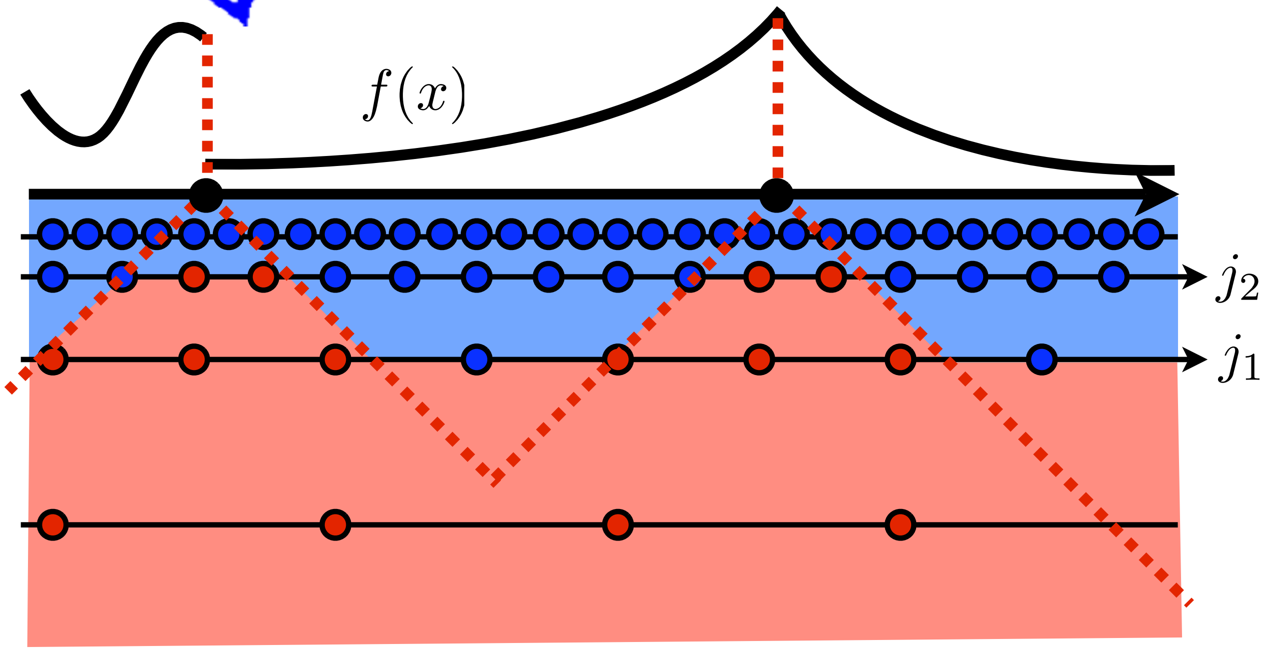
$$\|f - f_M\|^2 \leqslant \|f - \tilde{f}_M\|^2 \leqslant \sum_{j<j_2, n\in\mathcal{C}_j} |\langle f,\, \psi_{j,n}\rangle|^2 + \sum_{j<j_1, n\in\mathcal{C}_j^c} |\langle f,\, \psi_{j,n}\rangle|^2$$

$$\leqslant \sum_{j<j_2} (K|\mathcal{S}|) \times C^2 2^j + \sum_{j<j_1} 2^{-j} \times C^2 2^{j(2\alpha+1)}$$

$$= O(2^{j_2} + 2^{2\alpha j_1}) = O(T^2 + T^{\frac{2\alpha}{\alpha+1/2}}) = O(T^{\frac{2\alpha}{\alpha+1/2}})$$

Singulatities
$\ll$
regular part

Computing Error and #Coefficients



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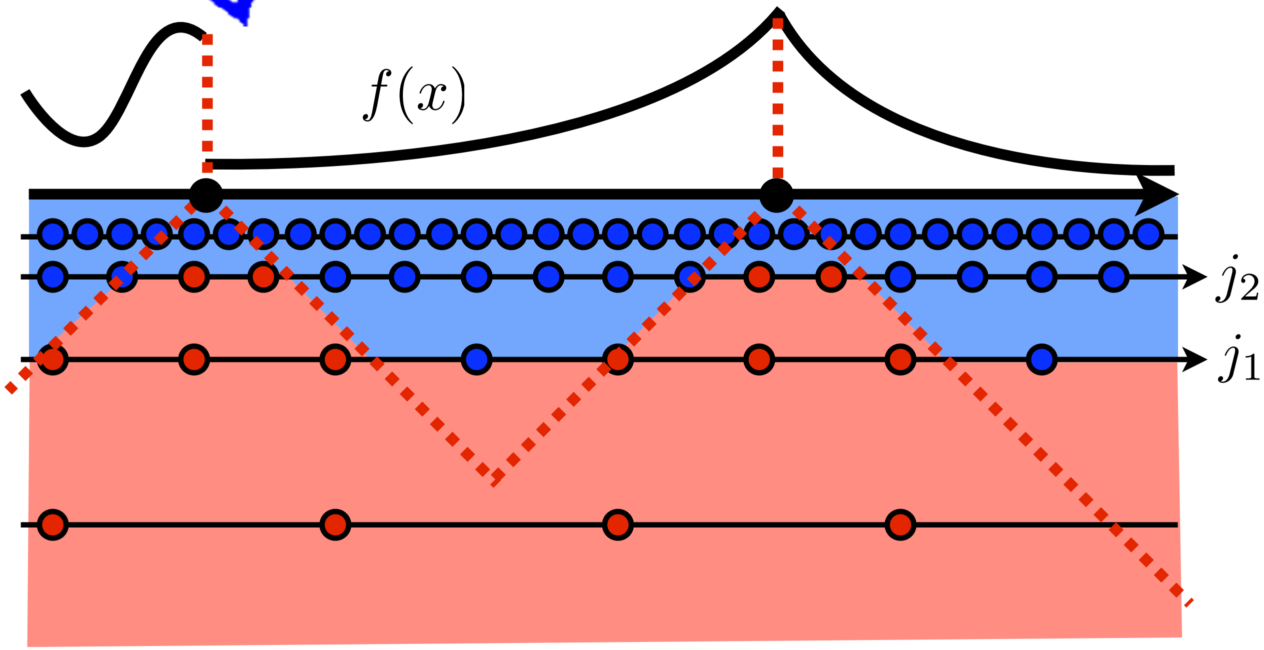
Singulatities $\ll$ regular part

$$M \leqslant \sum_{j\geqslant j_2} |\mathcal{C}_j| + \sum_{j\geqslant j_1} |\mathcal{C}_j^c| \leqslant \sum_{j\geqslant j_2} K|\mathcal{S}| + \sum_{j\geqslant j_1} 2^{-j}$$

$$= O(|\log(T)| + T^{\frac{-1}{\alpha+1/2}}) = O(T^{\frac{-1}{\alpha+1/2}})$$

Computing Error and #Coefficients



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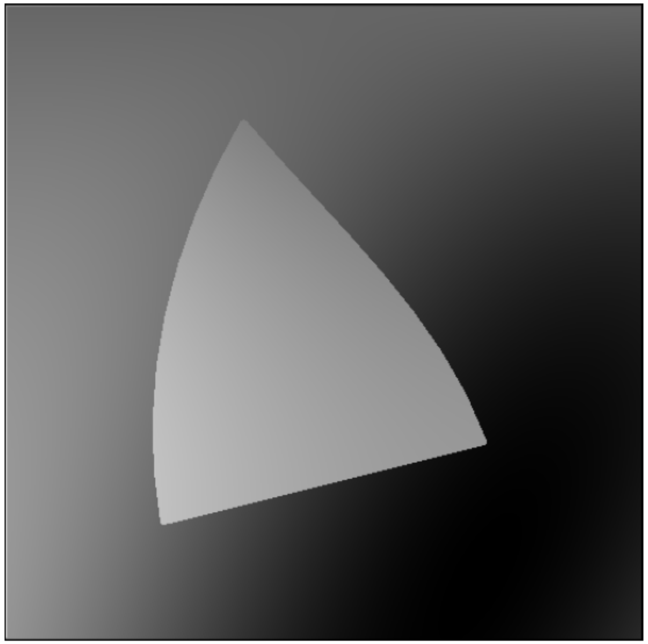
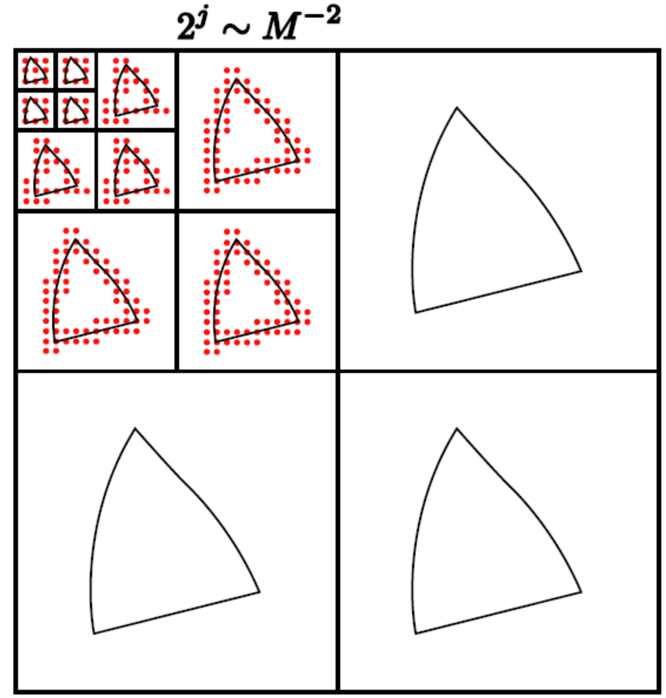
$$= O(|\log(T)| + T^{\frac{-1}{\alpha+1/2}}) = \boxed{O(T^{\frac{-1}{\alpha+1/2}})}$$

$$\boxed{\|f - f_M\|^2 = O(M^{-2\alpha})}$$

2D Wavelet Approximation





If f is C^α in $supp(\psi_{j,n})$, then

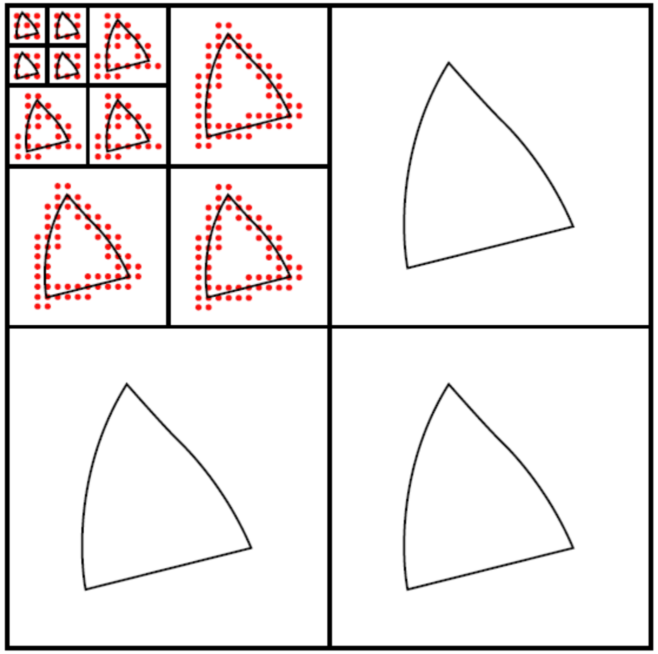
$$|\langle f,\, \psi_{j,n}\rangle| \leqslant 2^{j(\alpha+1)} \|f\|_{\mathrm{C}^\alpha} \|\psi\|_1$$

If f is bounded (e.g. around a singularity), then

$$|\langle f,\, \psi_{j,n}\rangle| \leqslant 2^{j} \|f\|_{\infty} \|\psi\|_1$$

Piecewise Regular Functions in 2D

$2^j \sim M^{-2}$



Theorem: If f is C^α outside a set of finite length edge curves,

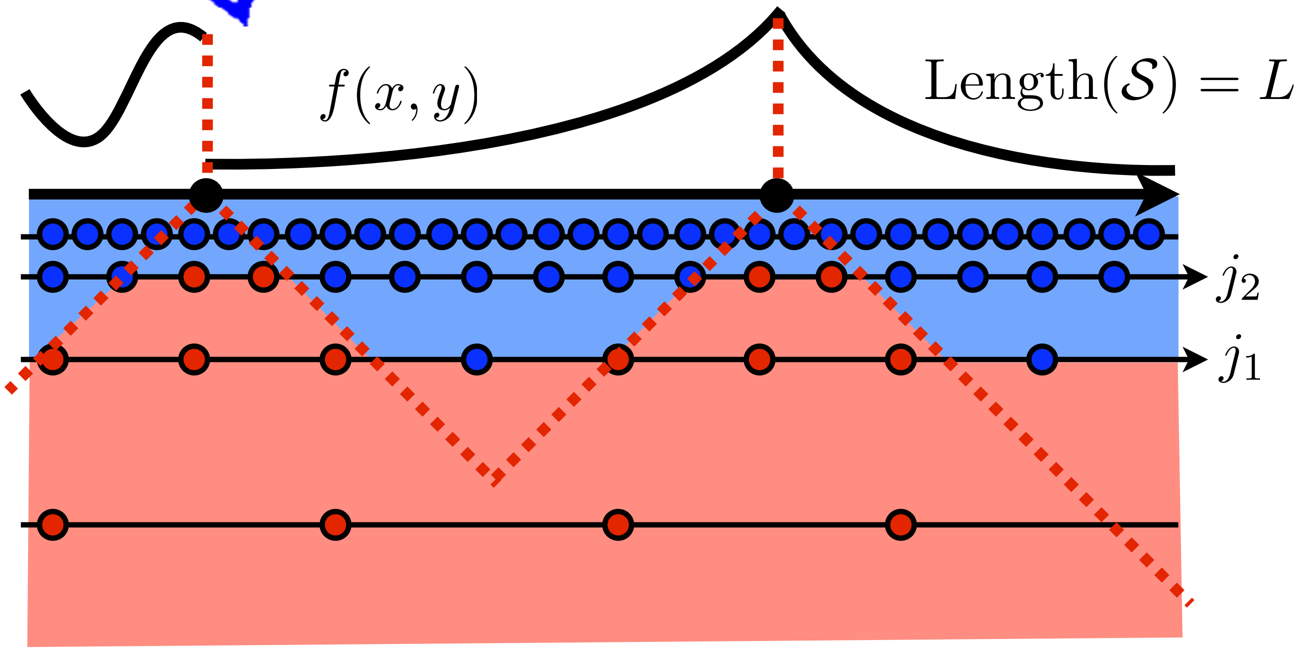
$$\varepsilon^n[M] = \|f - f_M^n\|^2 = \begin{cases} O(M^{-1/2}) & \text{(Fourier)}, \\ O(M^{-1}) & \text{(wavelets)}. \end{cases}$$

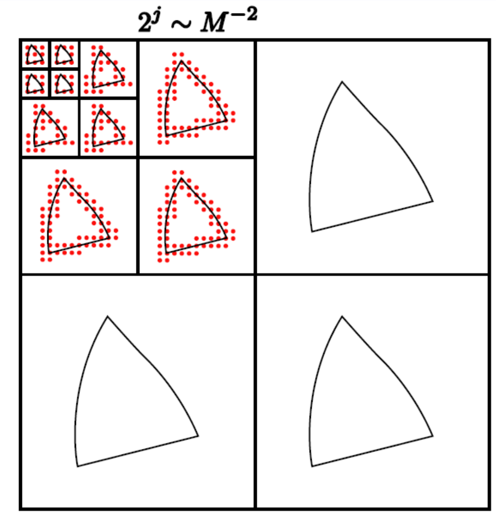
Fourier $\ll$ Wavelet, both sub-optimal.

Wavelets: same result for BV functions (optimal).

Example of 2D Approximations

Localizing the Singular Support in 2D

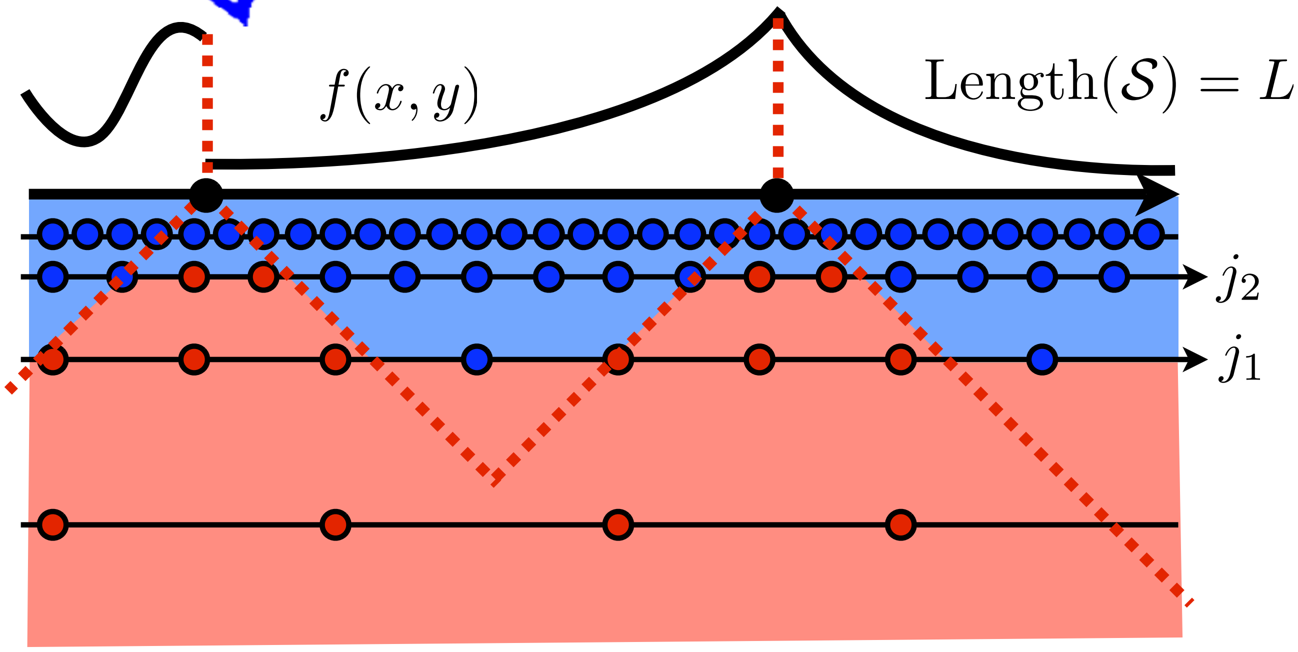


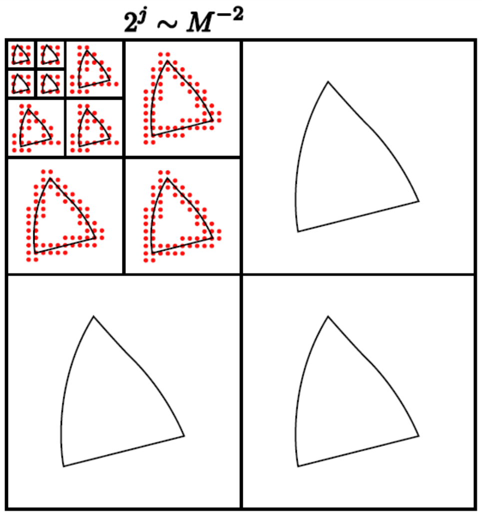


$$\mathcal{C}_j = \{n \setminus \mathrm{supp}(\psi_{j,n}) \cap \mathcal{S} \neq \emptyset\}$$

$$|\mathcal{C}_j| \leqslant LK2^{-j} \neq \text{constant}$$

Localizing the Singular Support in 2D





$$\mathcal{C}_j = \{n \setminus \mathrm{supp}(\psi_{j,n}) \cap \mathcal{S} \neq \emptyset\}$$

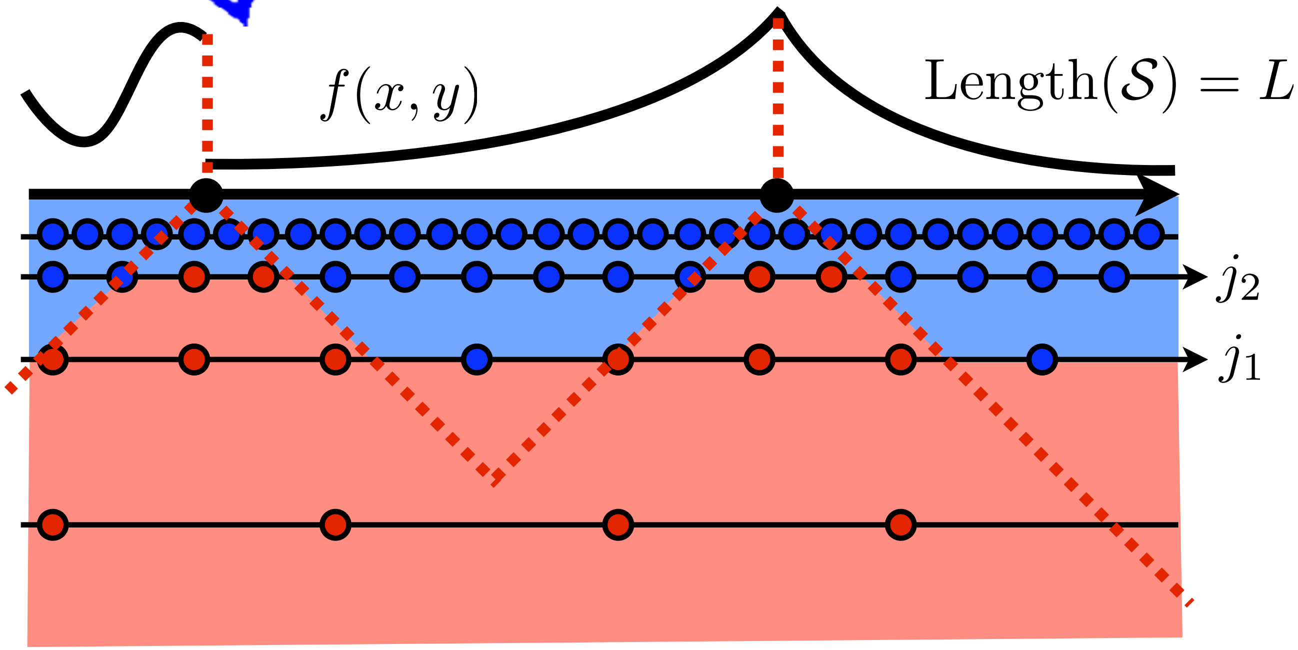
$$|\mathcal{C}_j| \leqslant LK2^{-j} \neq \text{constant}$$

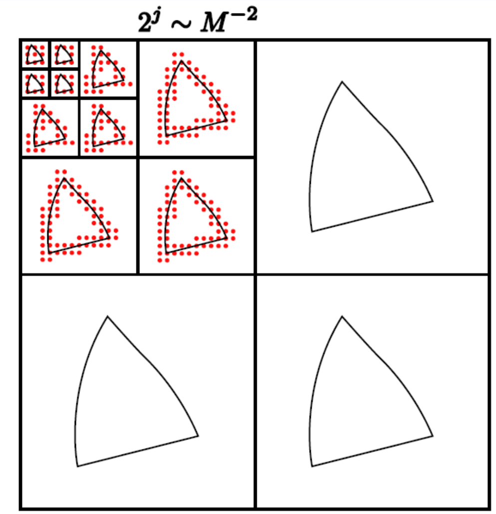
Coefficient behavior:

Regular, $n \in \mathcal{C}_j^c$: $\quad |\langle f, \psi_{j,n} \rangle| \leqslant C2^{j(\alpha+1)}$

Singular, $n \in \mathcal{C}_j$: $\quad |\langle f, \psi_{j,n} \rangle| \leqslant C2^{j}$

Localizing the Singular Support in 2D





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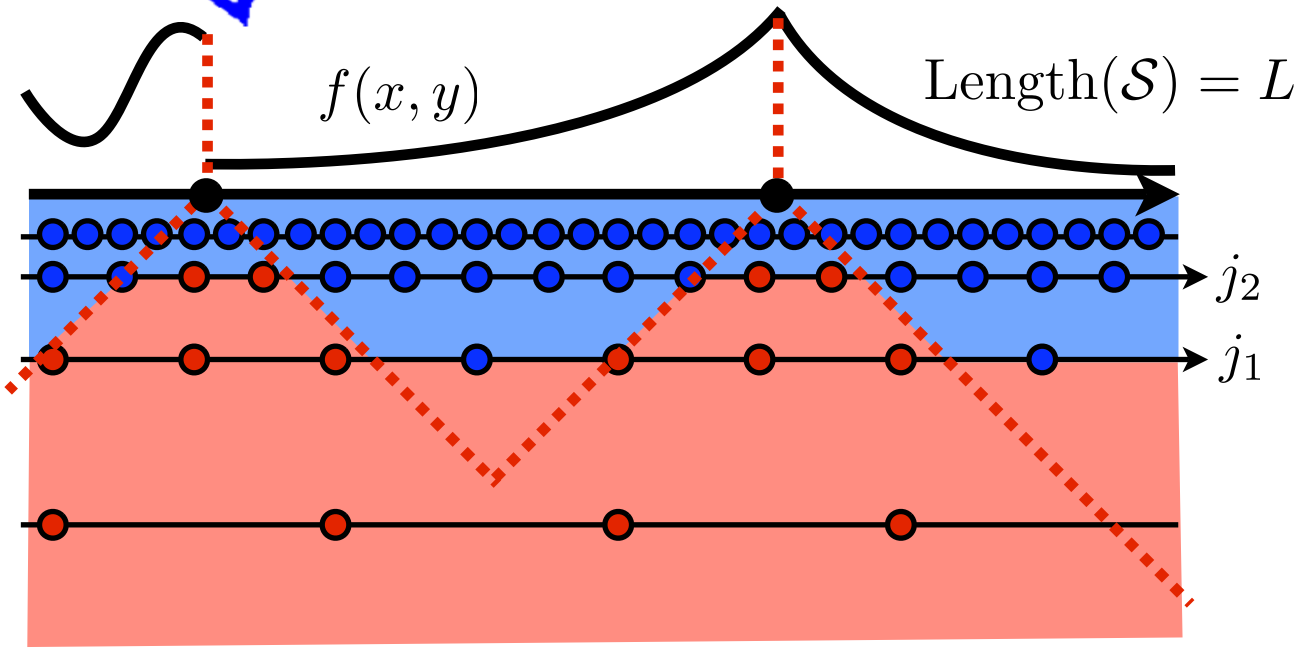
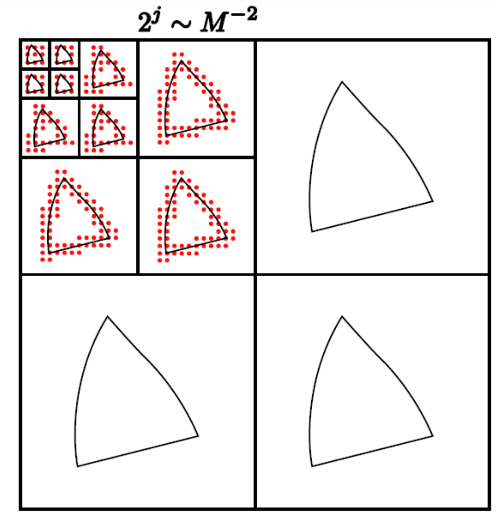
Cut-off scales (depends on T):

Singular: $\quad 2^{j_1} = (T/C)^{\frac{1}{\alpha+1}}$

Regular: $\quad 2^{j_2} = T/C$

Localizing the Singular Support in 2D





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Cut-off scales (depends on T):

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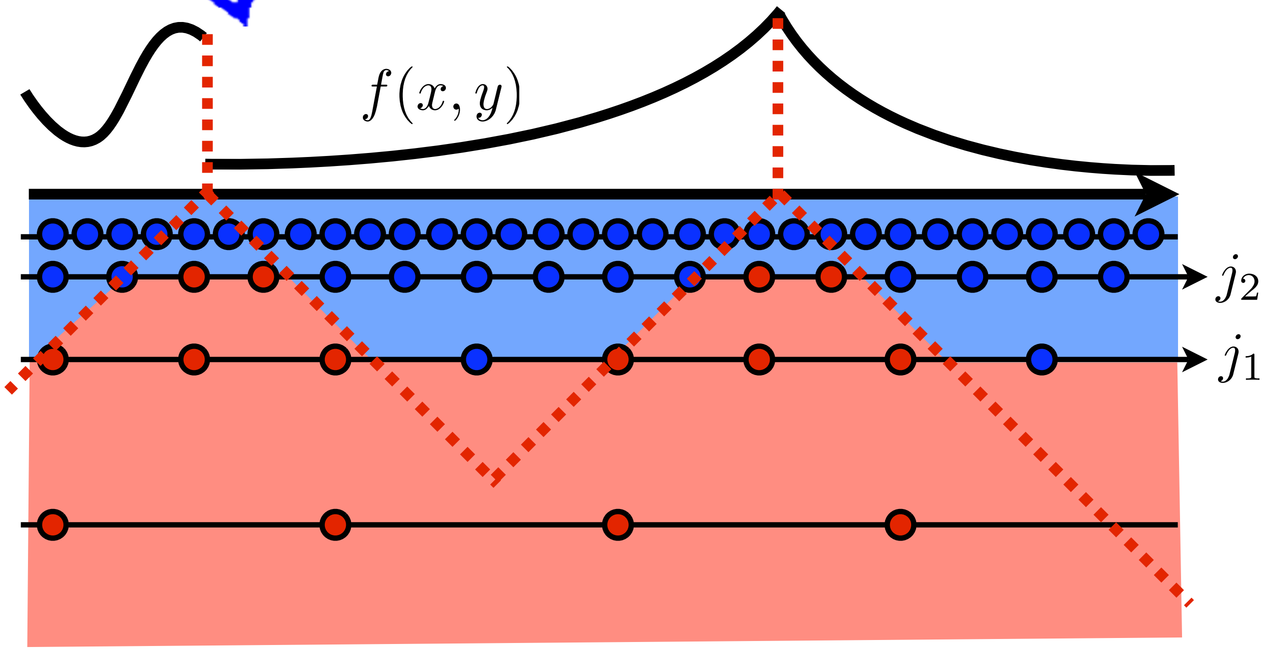
Hand-made approximate:

$$\tilde{f}_M = \sum_{j \geqslant j_2} \sum_{n \in \mathcal{C}_j} \langle f,\, \psi_{j,n}\rangle \psi_{j,n} + \sum_{j \geqslant j_1} \sum_{n \in \mathcal{C}_j^c} \langle f,\, \psi_{j,n}\rangle \psi_{j,n}$$

Computing Error and #Coefficients



$$|\mathcal{C}_j| \leqslant LK2^{-j} \neq \text{constant}$$

$$n \in \mathcal{C}_j^c: \ |\langle f,\, \psi_{j,n}\rangle| \leqslant C2^{j(\alpha+1)}$$

$$n \in \mathcal{C}_j: \ |\langle f,\, \psi_{j,n}\rangle| \leqslant C2^{j}$$

$$2^{j_1} = (T/C)^{\frac{1}{\alpha+1}}$$

$$2^{j_2} = T/C$$

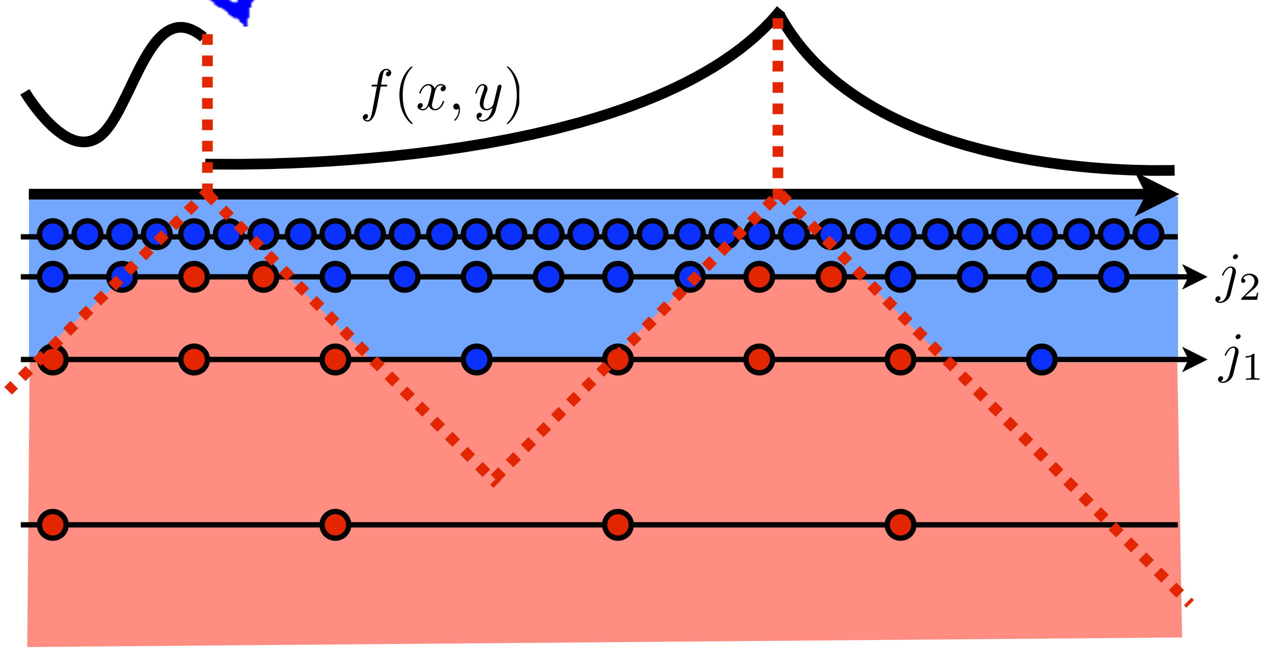
$$\|f - f_M\|^2 \leqslant \|f - \tilde{f}_M\|^2 \leqslant \sum_{j<j_2, n\in\mathcal{C}_j} |\langle f,\, \psi_{j,n}\rangle|^2 + \sum_{j<j_1, n\in\mathcal{C}_j^c} |\langle f,\, \psi_{j,n}\rangle|^2$$

$$\leqslant \sum_{j<j_2} LK2^{-j} \times C^2 2^{2j} + \sum_{j<j_1} 2^{-2j} \times C^2 2^{2j(\alpha+1)}$$

$$= O(2^{j_2} + 2^{2\alpha j_1}) = O(T + T^{\frac{2\alpha}{\alpha+1}}) = O(T)$$

Singulatities $\gg$ regular part

Computing Error and #Coefficients



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$$\|f - f_M\|^2 \leqslant \|f - \tilde{f}_M\|^2 \leqslant \sum_{j<j_2, n\in\mathcal{C}_j} |\langle f,\, \psi_{j,n}\rangle|^2 + \sum_{j<j_1, n\in\mathcal{C}_j^c} |\langle f,\, \psi_{j,n}\rangle|^2$$

$$\leqslant \sum_{j<j_2} LK2^{-j} \times C^2 2^{2j} + \sum_{j<j_1} 2^{-2j} \times C^2 2^{2j(\alpha+1)}$$

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Singulatities
$\gg$
regular part

$$M \leqslant \sum_{j\geqslant j_2} |\mathcal{C}_j| + \sum_{j\geqslant j_1} |\mathcal{C}_j^c| \leqslant \sum_{j\geqslant j_2} LK2^{-j} + \sum_{j\geqslant j_1} 2^{-2j}$$

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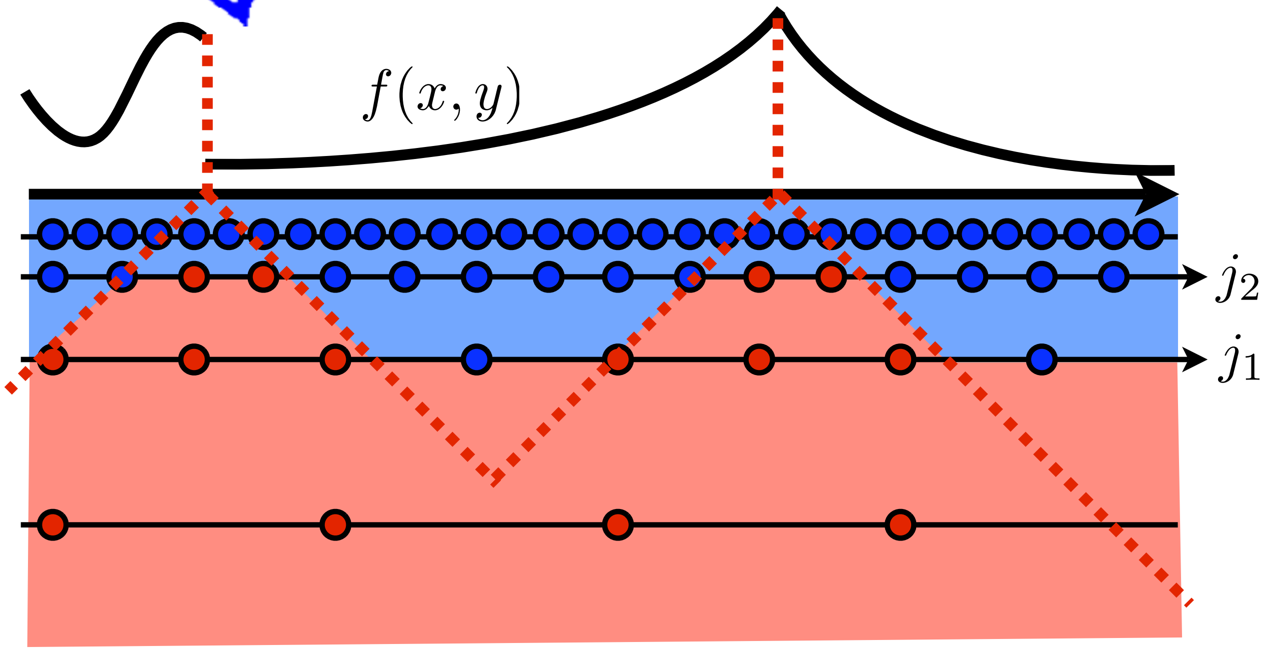
Computing Error and #Coefficients



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$$\|f - f_M\|^2 \leqslant \|f - \tilde{f}_M\|^2 \leqslant \sum_{j<j_2, n\in\mathcal{C}_j} |\langle f,\, \psi_{j,n}\rangle|^2 + \sum_{j<j_1, n\in\mathcal{C}_j^c} |\langle f,\, \psi_{j,n}\rangle|^2$$

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Singulatities $\gg$ regular part

$$M \leqslant \sum_{j\geqslant j_2} |\mathcal{C}_j| + \sum_{j\geqslant j_1} |\mathcal{C}_j^c| \leqslant \sum_{j\geqslant j_2} LK2^{-j} + \sum_{j\geqslant j_1} 2^{-2j}$$

$$= O(T^{-1} + T^{\frac{-1}{\alpha+1}}) = \boxed{O(T^{-1})}$$

$$\boxed{\|f - f_M\|^2 = O(M^{-1})}$$

Overview

- Approximation and Compression
- Decay of Approximation Error
- Fourier for Smooth Functions
- Wavelet for Piecewise Smooth Functions
- Curvelets and Finite Elements for Cartoons

Geometrically Regular Images

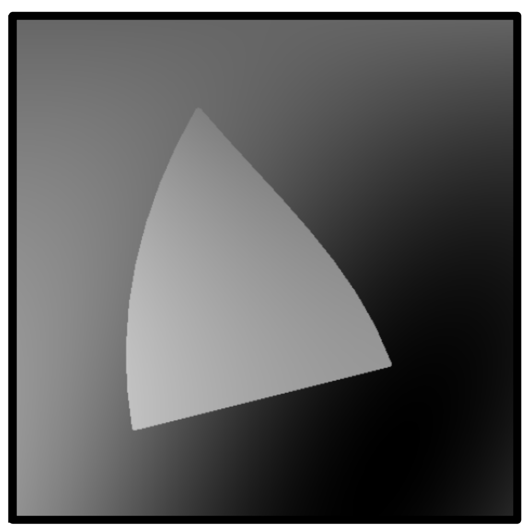
Geometric image model: f is C^α outside a set of C^α edge curves.

BV image: level sets have finite lengths.

Geometric image: level sets are regular.

$\int |\nabla f|$

Geometry = cartoon image



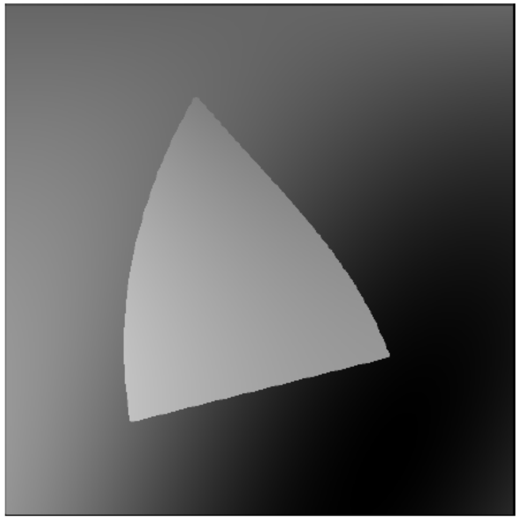
Sharp edges

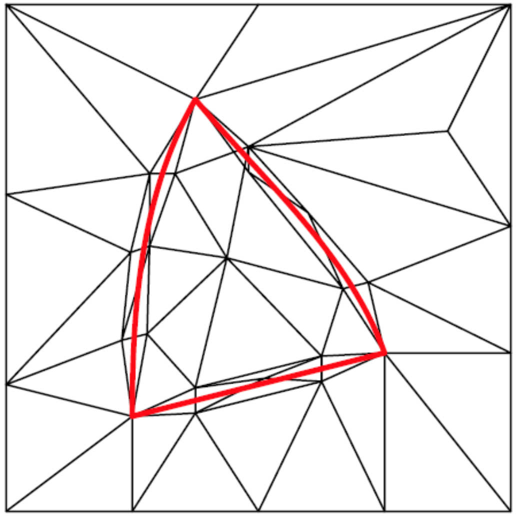
Smoothed edges

Geometic Construction : Finite Elements

Approximation of f, C^2 outside C^2 edges.

Piecewise linear approximation on M triangles: $\tilde{f}_M$.





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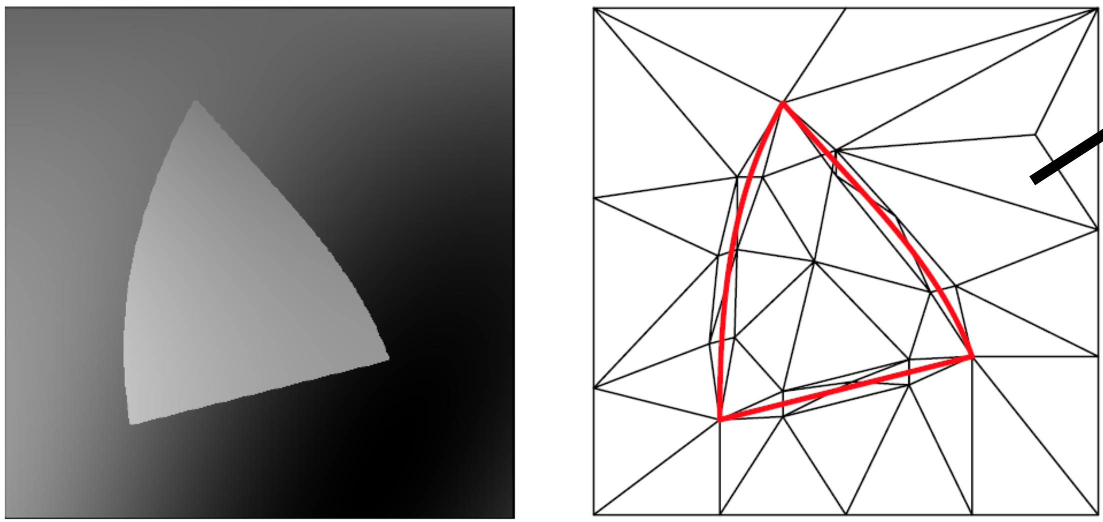
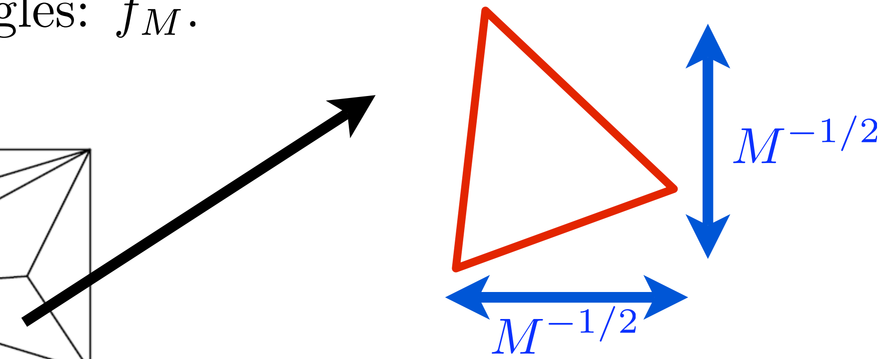


Regular areas:

$\sim M/2$ equilateral triangles.



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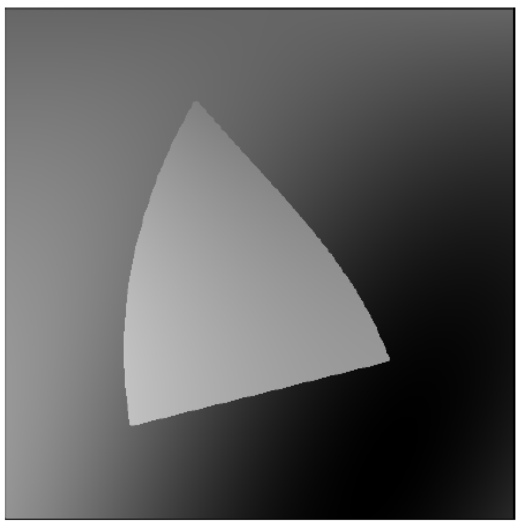

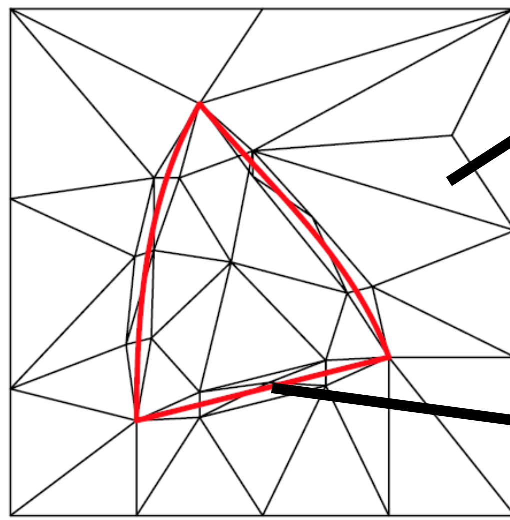

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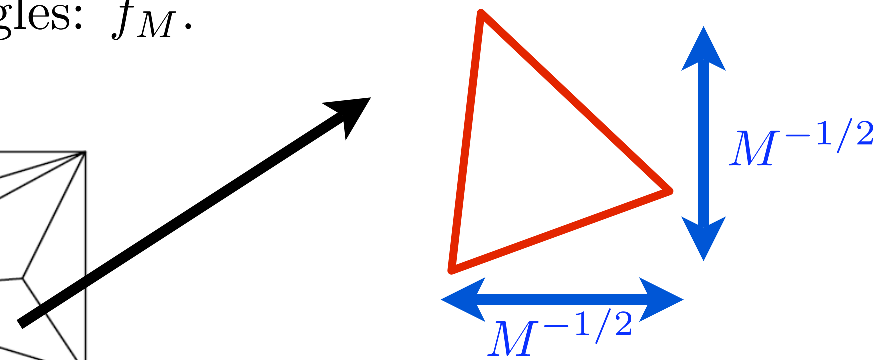
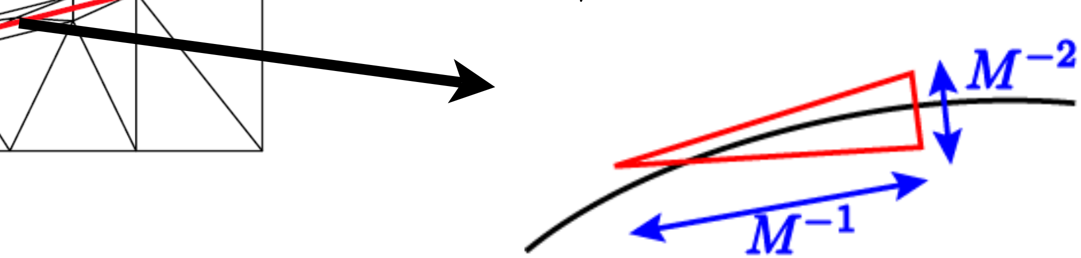


Singular areas:

$\sim M/2$ anisotropic triangles.



Geometic Construction : Finite Elements

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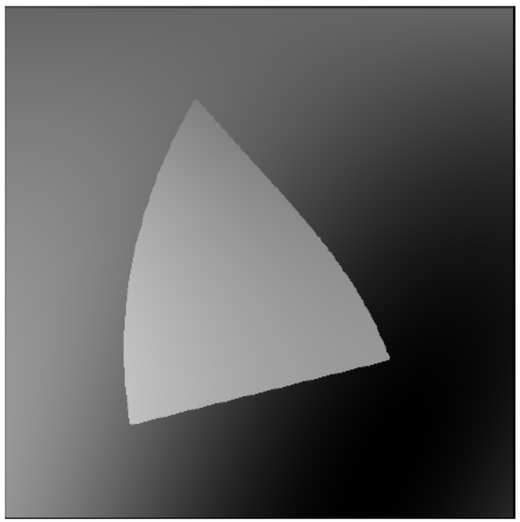

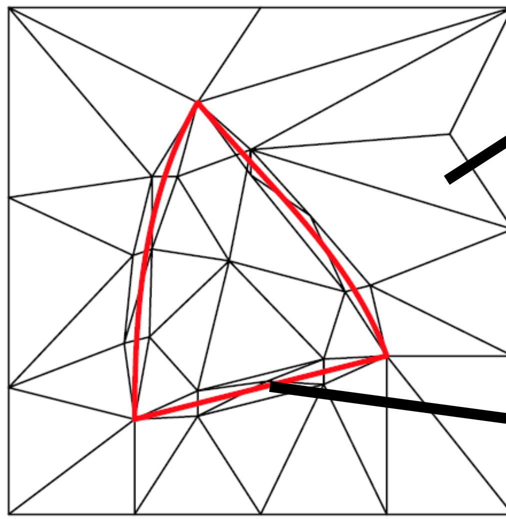

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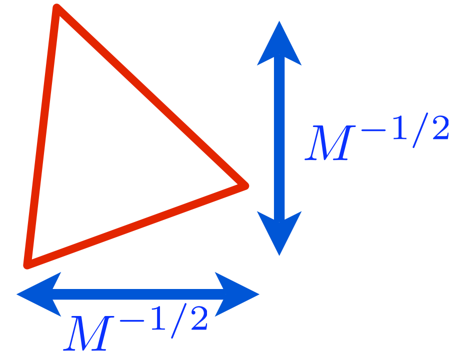

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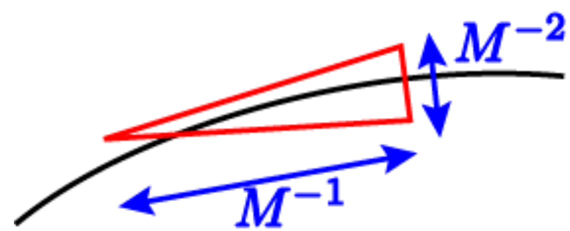


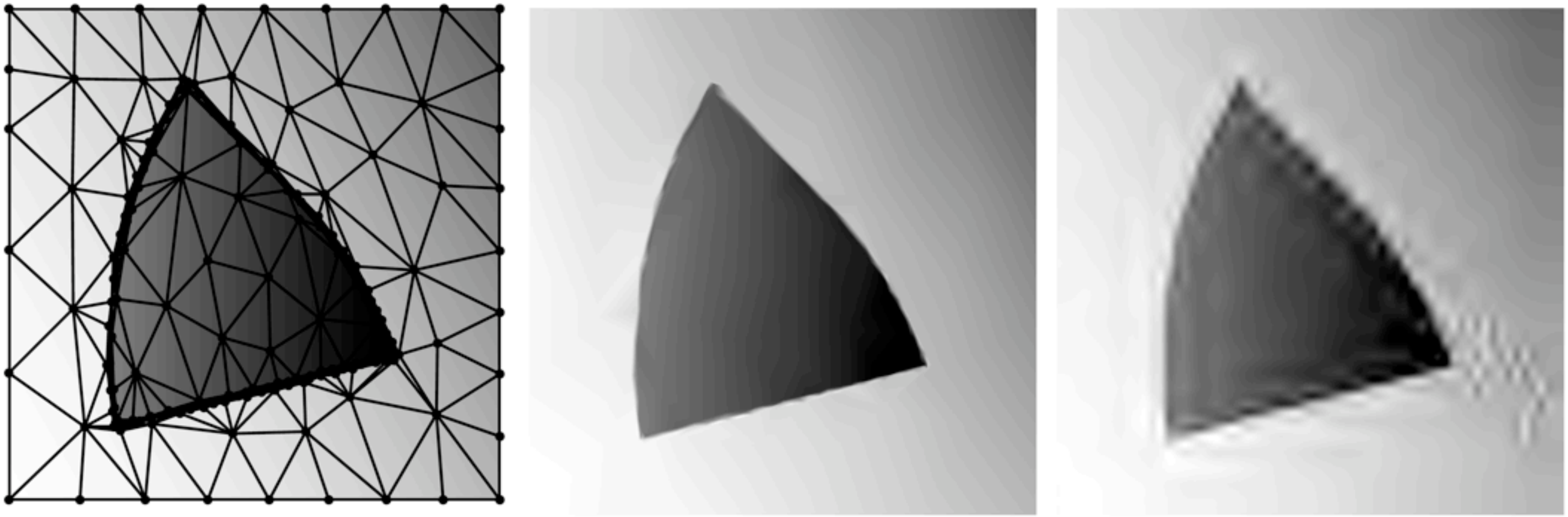
Theorem: If f is C^2 outside a set of C^2 contours, then one has for an adapted triangulation $\|f - \tilde{f}_M\|^2 = O(M^{-2})$.

Difficulties to build efficient approximations.

No optimal strategies (greedy solutions).

Greedy Triangulation Optimization

Bougleux, Peyré, Cohen, ECCV'08



anisotropic, SNR=29.9dB   JPEG-2000, SNR=24.2dB

anisotropic, SNR=21.9dB   JPEG-2000, SNR=23.2dB

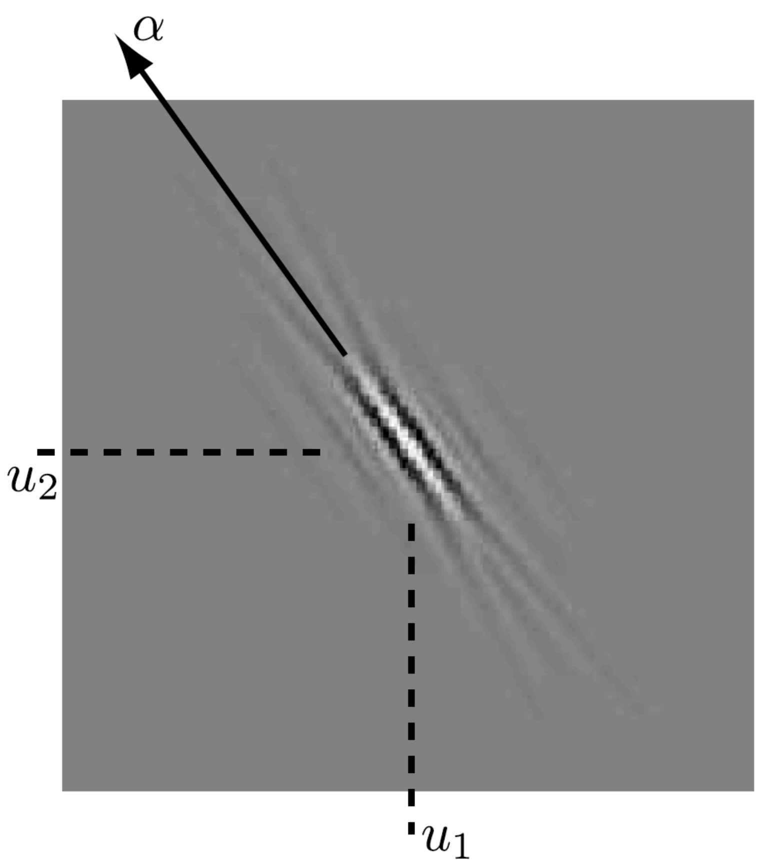
Curvelet Atoms

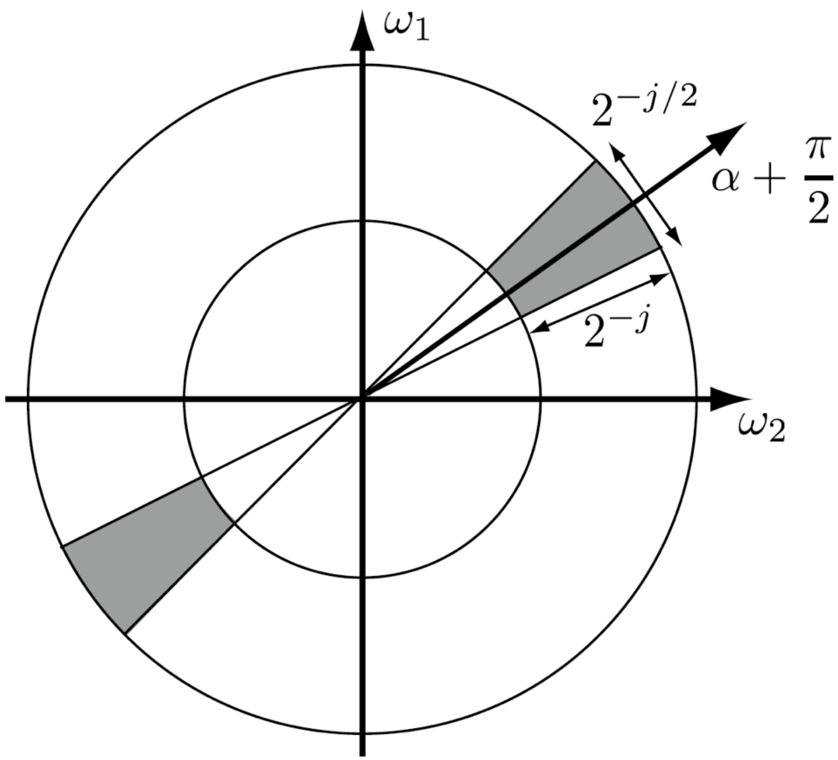
[Candes, Donoho] [Candes, Demanet, Ying, Donoho]

Parabolic dyadic scaling: $c_{2^j}(x_1, x_2) \approx 2^{-3j/4}\, c(2^{-j/2}x_1, 2^{-j}x_2).$

Rotation: $c^{\theta}_{2^j,u}(x_1, x_2) = c_{2^j}(R_\theta(x-u))$

“width $\approx$ *length*^{2}”





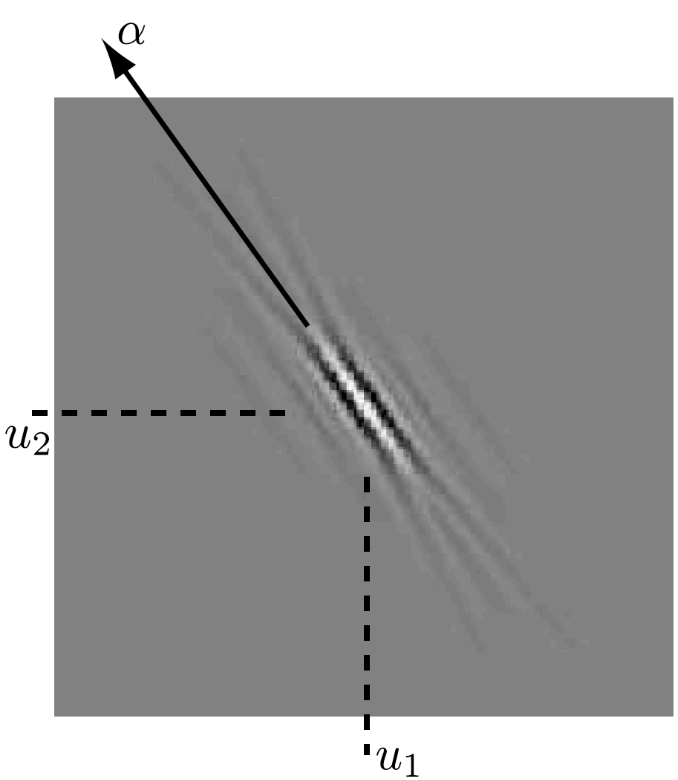
Curvelet Tight Frame

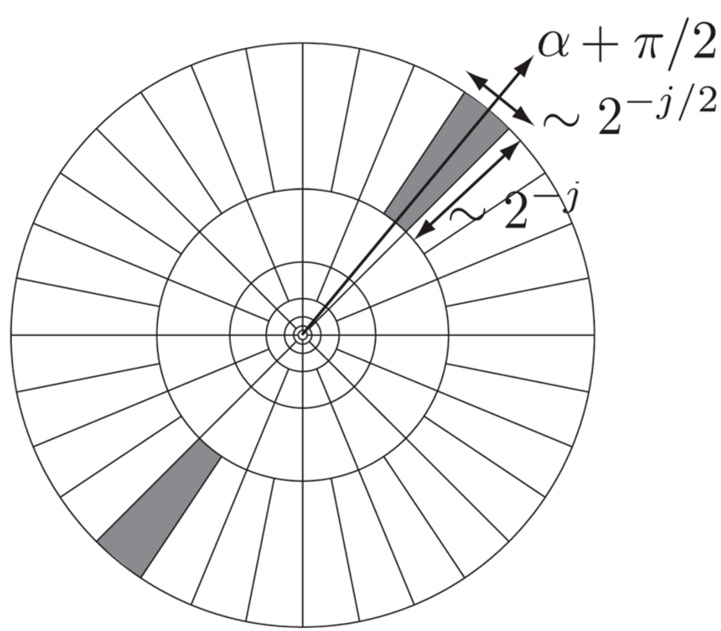
Angular sampling: $\Theta_j = \{\alpha = k\pi 2^{\lfloor j/2 \rfloor -1} \quad \text{for} \quad 0 \leqslant k < 2^{-\lfloor j/2 \rfloor +2}\}$

Spacial sampling: $\forall m = (m_1, m_2) \in \mathbb{Z}^2 \quad , \quad u_m^{(j,\alpha)} = R_\alpha(2^{j/2} m_1, 2^j m_2)$

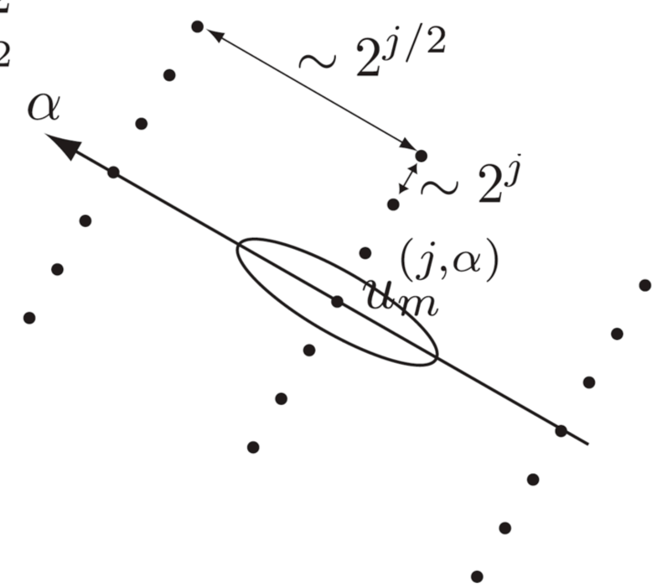
Tight frame of $\mathrm{L}^2(\mathbb{R}^2)$: $\left\{ c_{j,m}^{\alpha}(x) = c_{2^j}^{\alpha}\left(x - u_m^{(j,\alpha)}\right) \right\}_{j\in\mathbb{Z}, \alpha \in \Theta_j, m \in \mathbb{Z}^2}$.

$$\|f\|^2 = \sum_{j\in\mathbb{Z}} \sum_{\alpha\in\Theta_j} \sum_{m\in\mathbb{Z}^2} |\langle f,\, c_{j,m}^{\alpha}\rangle|^2 \qquad f(x) = \sum_{j\in\mathbb{Z}} \sum_{\alpha\in\Theta_j} \sum_{m\in\mathbb{Z}^2} \langle f,\, c_{j,m}^{\alpha}\rangle c_{j,m}^{\alpha}(x)$$





Frequency tiling



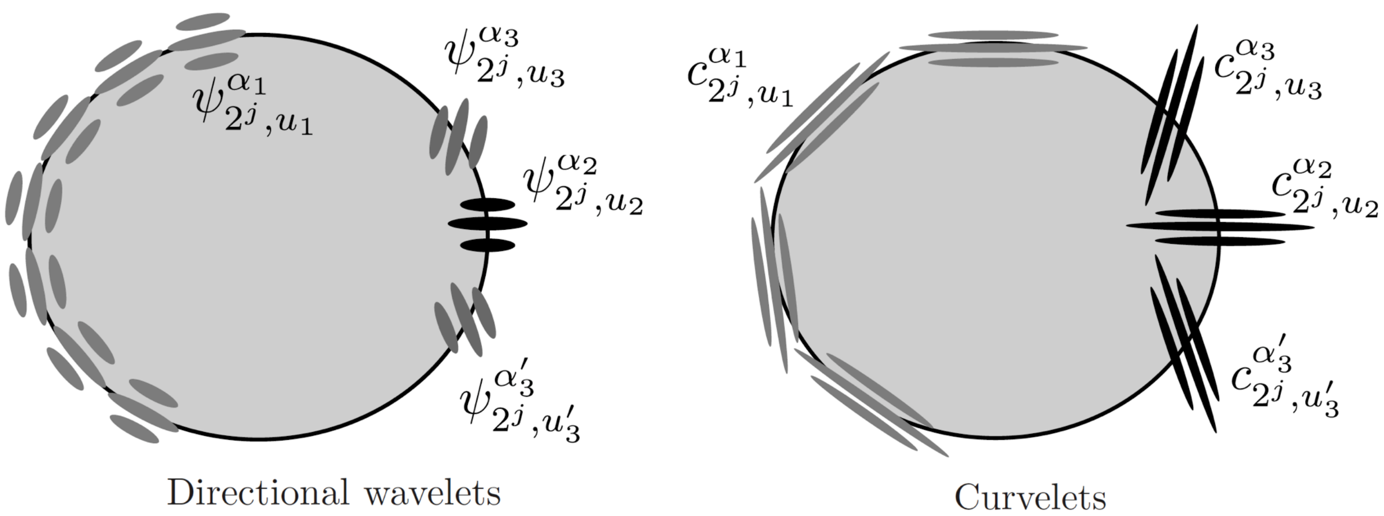
Spatial sampling

Curvelet Approximation

M-term curvelet approximation:

$$f_M = \sum_{(\theta,j,m)\in\Lambda_T} \langle f,\, c^{\theta}_{j,m}\rangle\, c^{\theta}_{j,m} \quad \text{with} \quad \Lambda_T = \{(j,\theta,m)\ :\ |\langle f,\, c^{\theta}_{j,m}\rangle| > T\}$$

Theorem: If f is C^2 outside a set of C^2 edges, $\|f - f_M\|^2 = O(M^{-2}(\log M)^3)$.

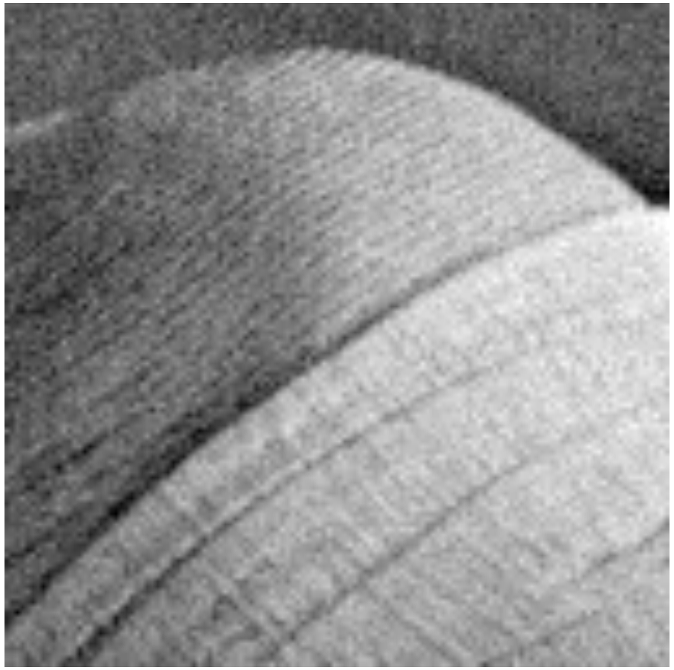


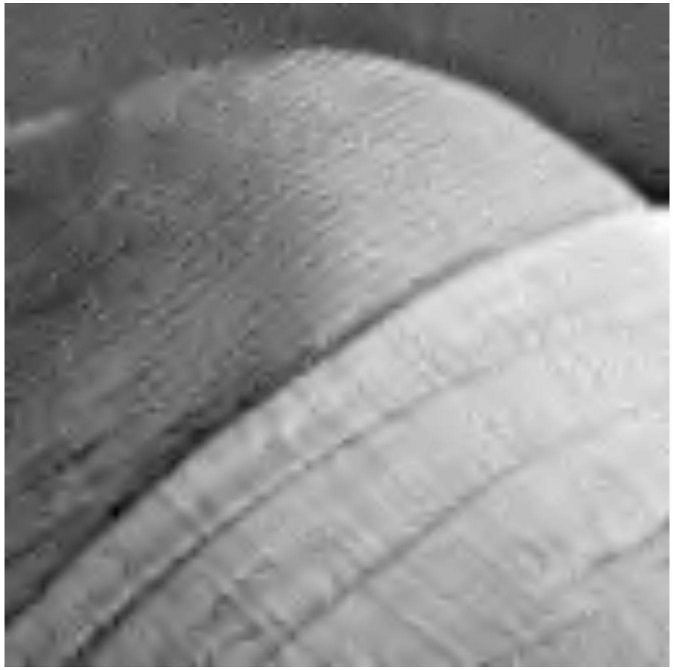
Discrete curvelets: $O(N\log(N))$ algorithm.

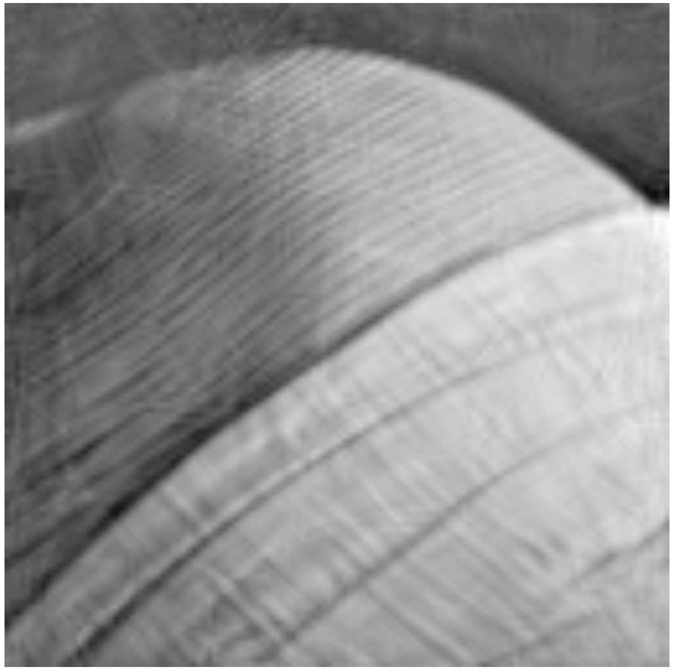
`www.curvelet.org`

Redundancy $\approx 5 \quad \Longrightarrow \quad$ not efficient for compression.

Curvelets Denoising


Noisy


Wavelets


Curvelets

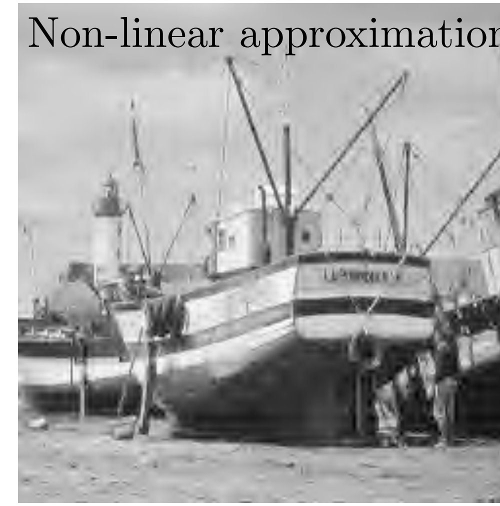
Works on elongated edges.

Works also on locally parallel textures !

Conclusion

Linear vs. non-linear approximation.

Transform coding $\approx$ *approximation.*



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JPEG-2000:

– Embedded coder.

– Statistical modeling of coefficients.

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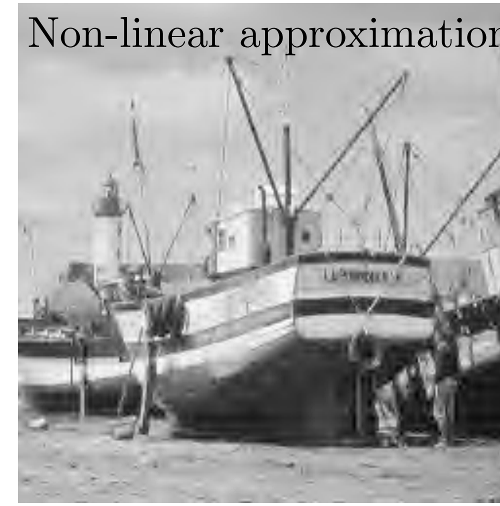
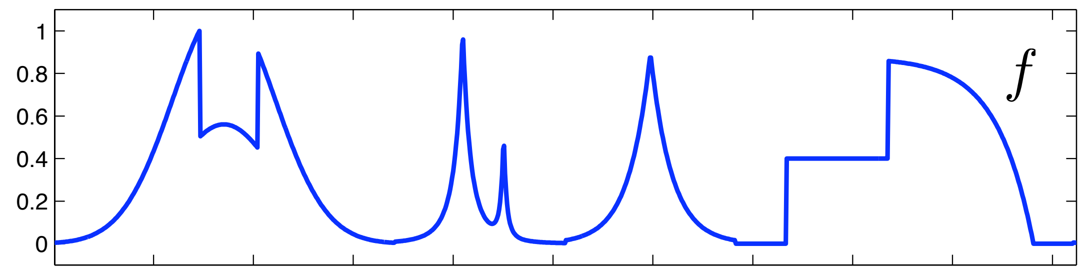
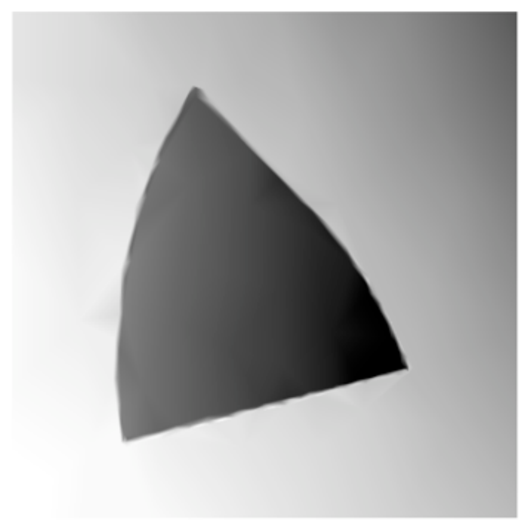
Wavelet approximation:

– Optimal for piecewise-smooth 1D signals.

– Sub-optimal for cartoon images.







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– Sub-optimal for cartoon images.

Improvements:

– Finite elements triangulations.

– Curvelets, wedgeprints, bandlets, etc.