Wavelet Processing

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http://www.ceremade.dauphine.fr/~peyre/numerical-tour/

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Overview

- Multiresolutions
- Fast Wavelet Transform
- Wavelet Design
- 2D Wavelet Processing

Multiresolutions: Approximation Spaces

Nested spaces $\{V_j\}_{j\in\mathbb{Z}}$ obtained by dilatation: $f(t) \in V_j \Leftrightarrow f(t/2) \in V_{j+1}$.

Stable under dyadic translation: $f(t) \in V_j \Leftrightarrow \forall n \in \mathbb{Z}, f(t + n2^j) \in V_j$

$$\mathrm{L}^2(\mathbb{R}) \longrightarrow \cdots \longrightarrow V_{j-1} \longrightarrow V_j \longrightarrow V_{j+1} \longrightarrow \cdots \longrightarrow \{0\}$$

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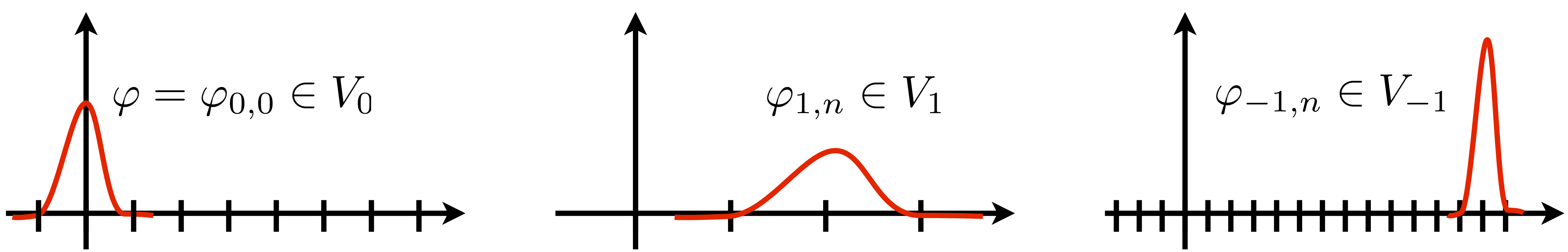
Scaling function φ: $\quad \mathrm{V}_j = \mathrm{Span}(\varphi_{j,n})_{n\in\mathbb{Z}} \quad \varphi_{j,n}(t) = \dfrac{1}{\sqrt{2^j}}\varphi\left(\dfrac{t-2^j n}{2^j}\right)$



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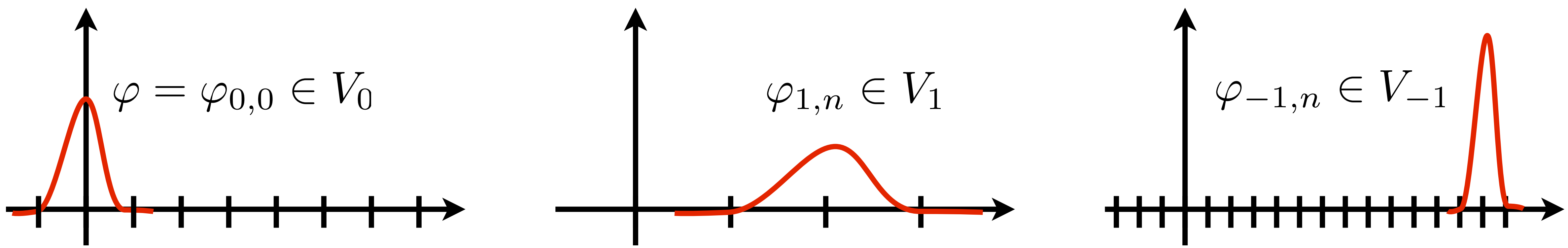

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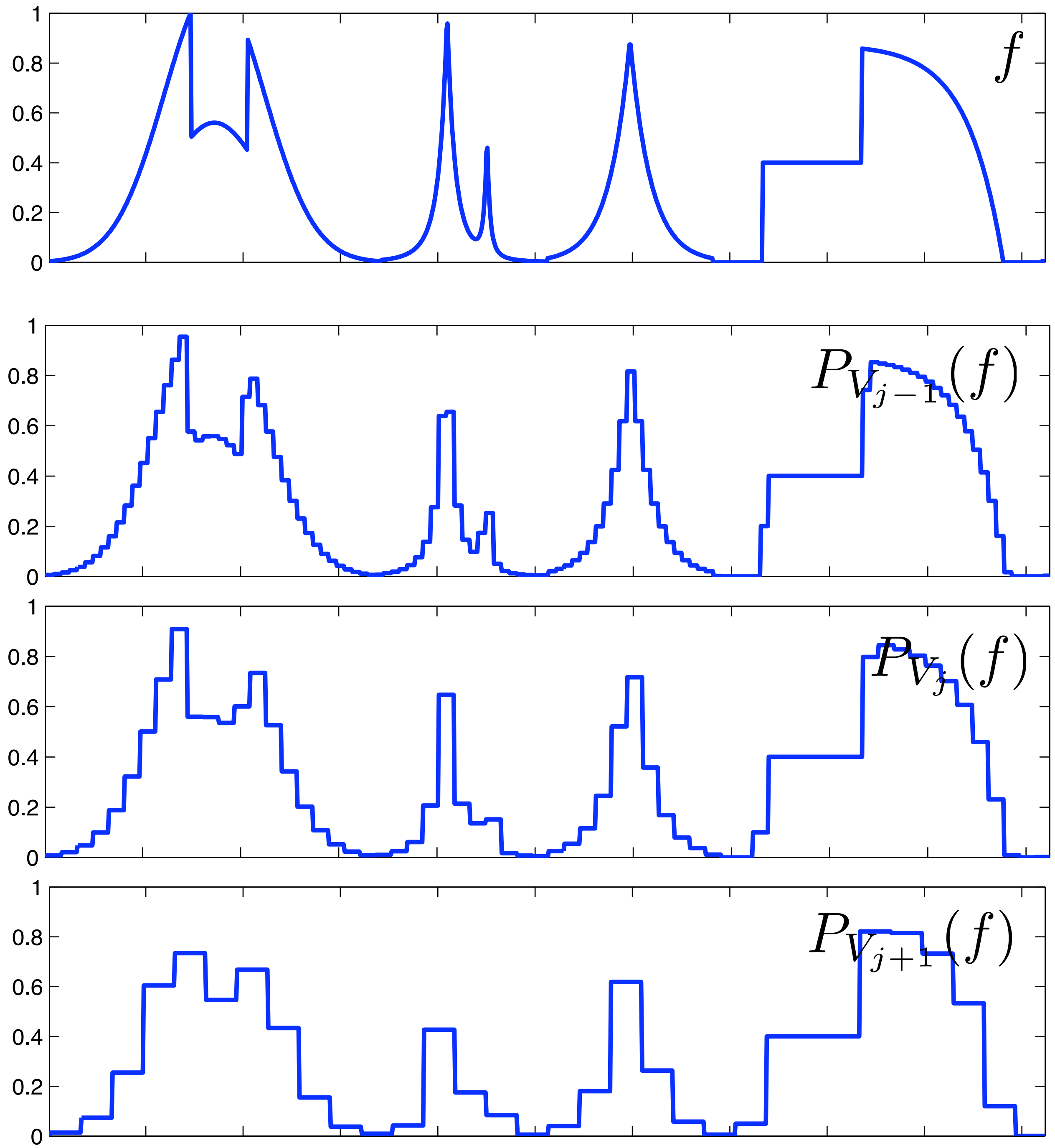
Bouded domains $f \in \mathrm{L}^2([0,1])$:

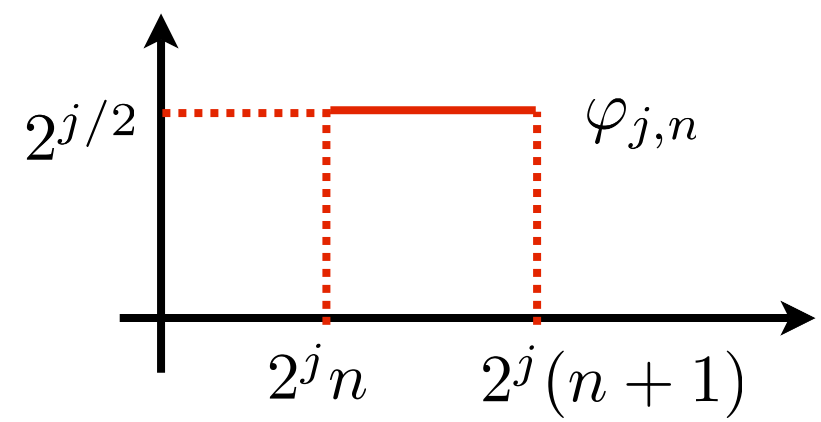
- Periodic boundary conditions: $[0,1] \simeq \mathbb{R}/\mathbb{Z}$, use only $j \leqslant 0$.
- Symmetric boundary conditions: non-translation invariant, more difficult.

Haar Multiresolutions

Piecewise-constant approximation: $\mathrm{V}_j = \{f \setminus f \text{ constant on } [2^j n, 2^j(n+1))\}$

Linear approximation of f using V_j: $P_{V_j}(f) = \sum_n \langle f, \varphi_{j,n} \rangle \varphi_{j,n}$





Generalizes to higher order spline approximations.

Multiresolutions: Detail Spaces

Detail space W_j: $\quad V_{j-1} = V_j \oplus^{\perp} W_j.$

$$\begin{tikzcd}
\mathrm{L}^2(\mathbb{R}) \arrow[r] & \cdots \arrow[r] \arrow[dr] & V_{j-1} \arrow[r] \arrow[dr] & V_j \arrow[r] \arrow[dr] & V_{j+1} \arrow[r] \arrow[dr] & \cdots \arrow[r] & \{0\} \\
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Decomposition into detail spaces: $\quad \mathrm{L}^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{+\infty} W_j = V_{j_0} \bigoplus_{j\leqslant j_0} W_j$

Wavelet basis of $\mathrm{L}^2(\mathbb{R})$: $\quad$ Full: $\left\{\psi_{j,n} \setminus (j,n) \in \mathbb{Z}^2\right\}$

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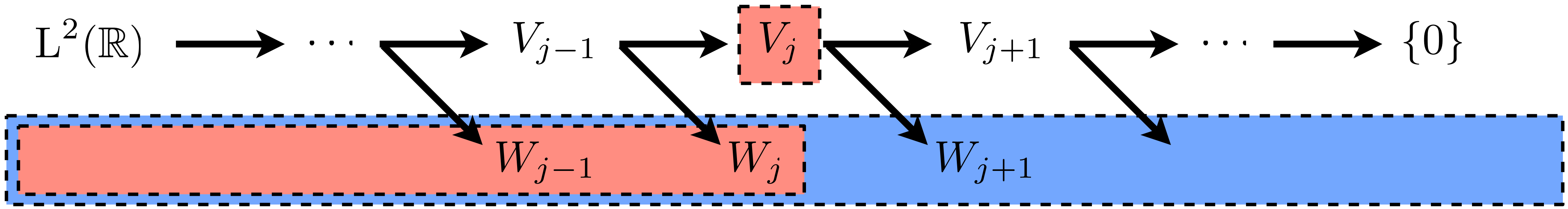

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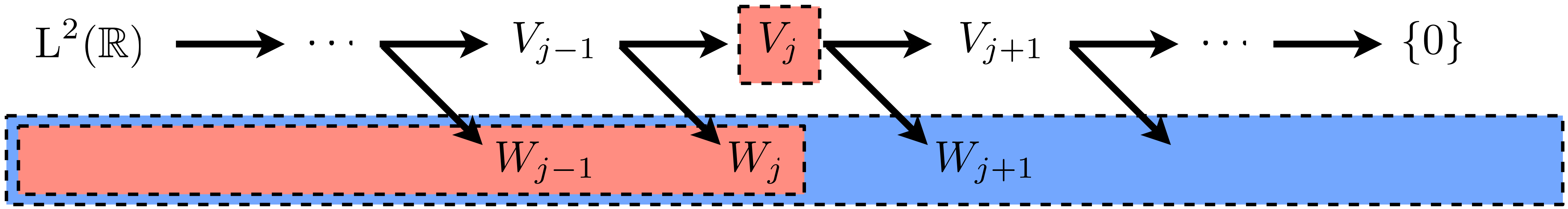

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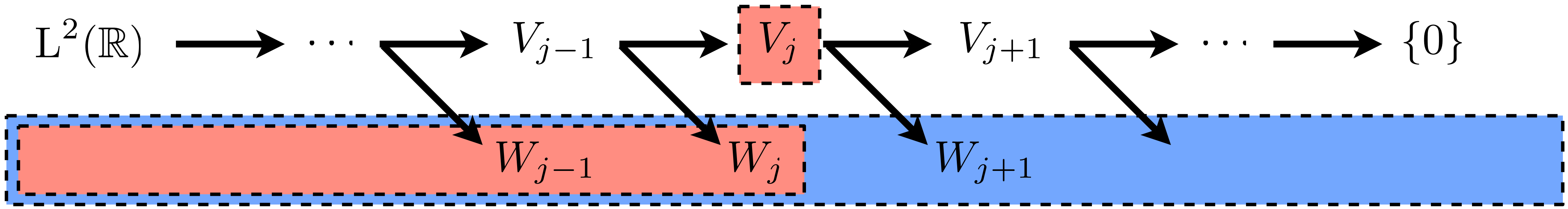

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Wavelet transform: computing the coefficients

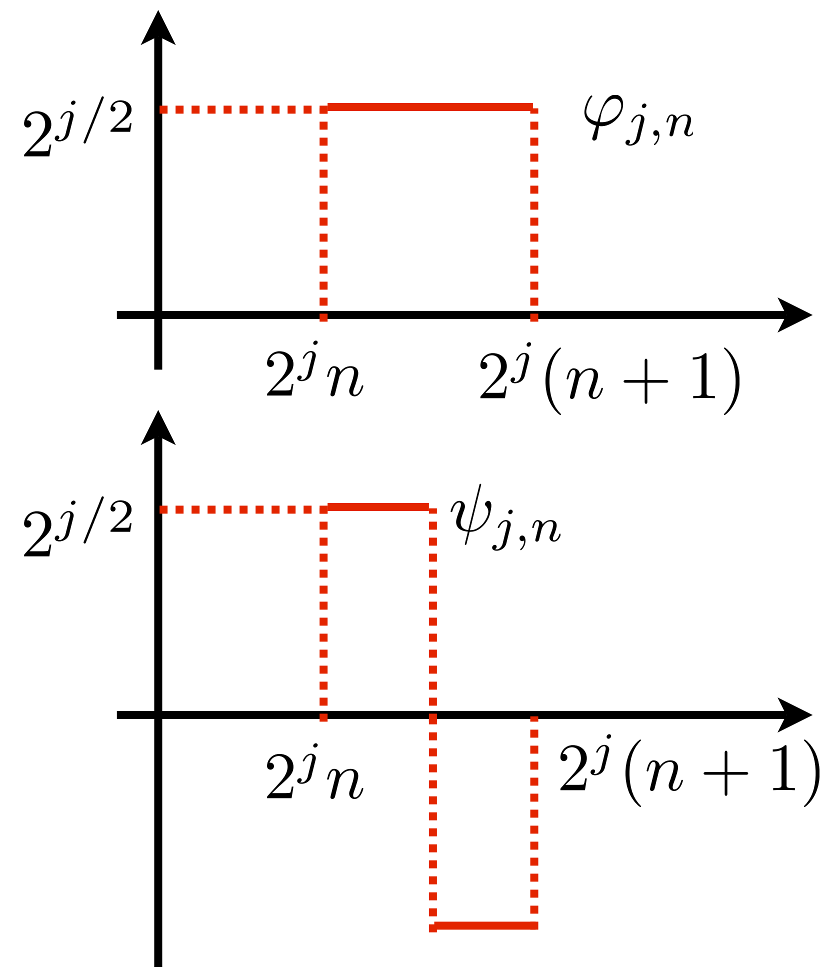
$$\{\langle f,\, \psi_{j,n}\rangle\}_{0\leqslant n<2^{-j}}^{j\leqslant j_0} \cup \{\langle f,\, \varphi_{j_0,n}\rangle\}_{0\leqslant n<2^{-j_0}}$$

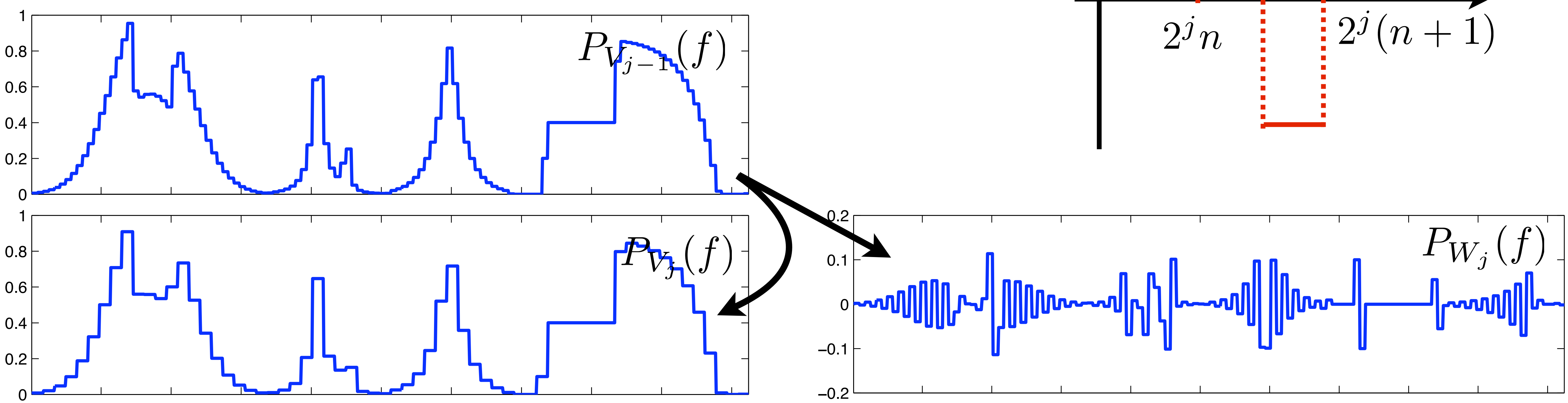
Haar Wavelets

Projection on approx. space: $P_{V_j}(f) = \sum_n \langle f,\, \varphi_{j,n} \rangle \varphi_{j,n}$

Projection on details space: $P_{W_j}(f) = \sum_n \langle f,\, \psi_{j,n} \rangle \psi_{j,n}$

$$\underbrace{P_{V_{j-1}}(f)}_{\text{Fine approximation}} = \underbrace{P_{V_j}(f)}_{\text{Coarse approximation}} + \underbrace{P_{W_j}(f)}_{\text{Details}}$$





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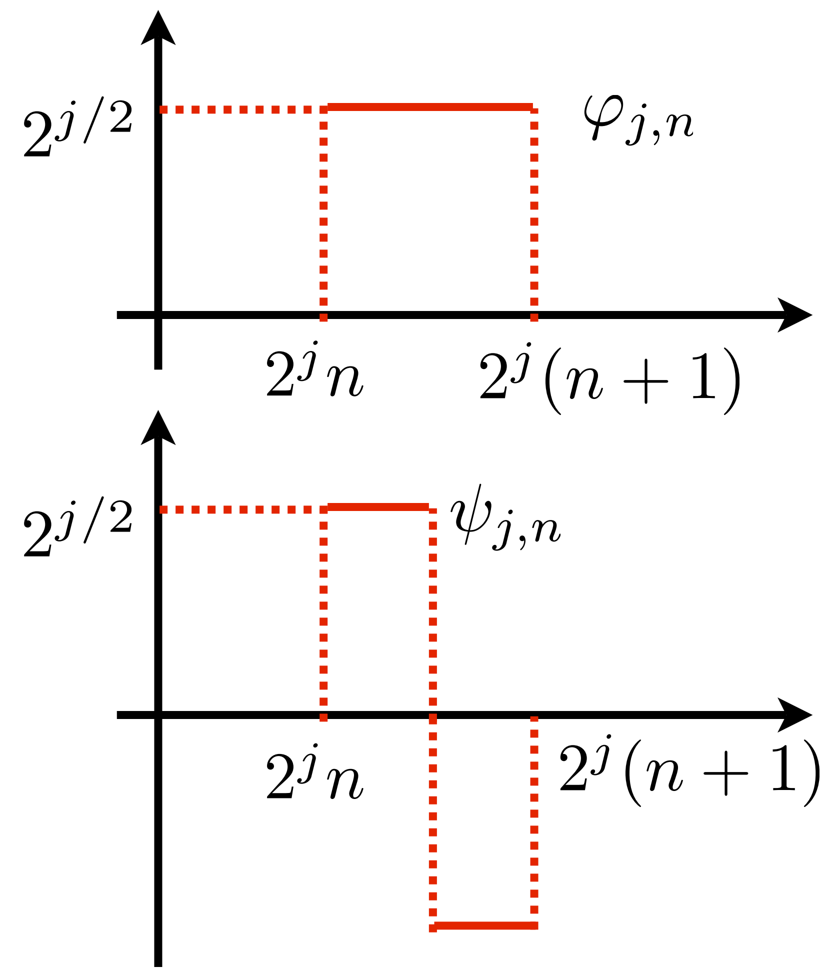
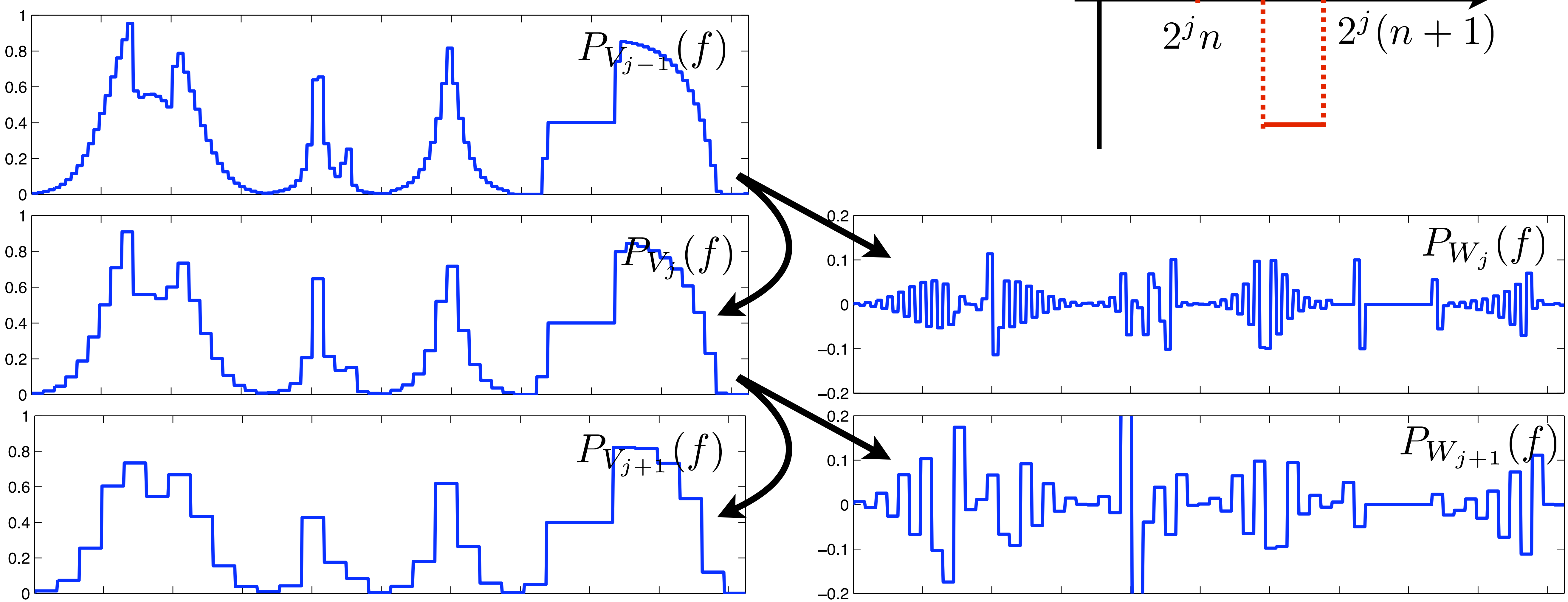
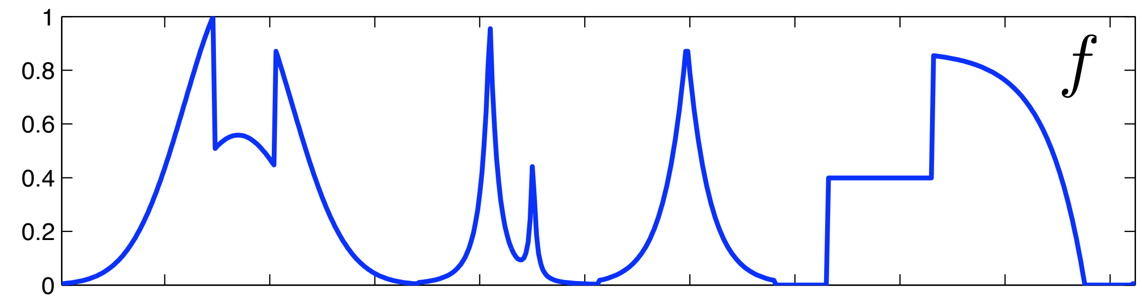




Discrete Wavelet Coefficients



Analog signal $f_0 \in \mathrm{L}^2([0,1])$.

Discretization points $\{n2^J\}_n$ where $2^{-J} = N$.

Discrete sampled signal:

$$f[n] = a_J[n] = \int f_0(t)\varphi(t/2^J - n)\mathrm{d}t = \langle f_0,\, \varphi_{J,n} \rangle$$

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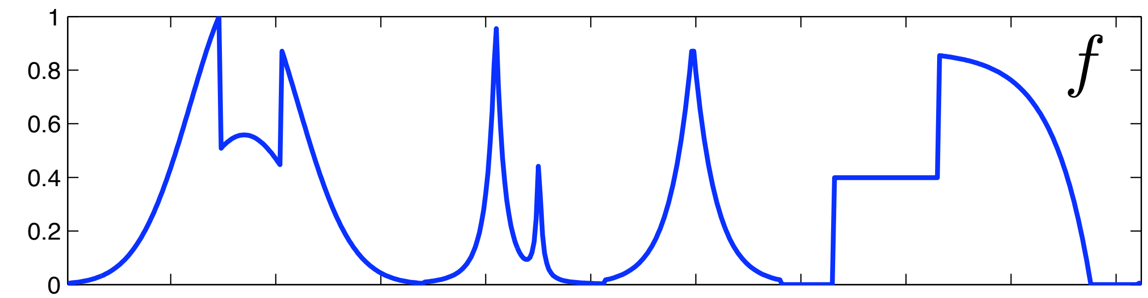

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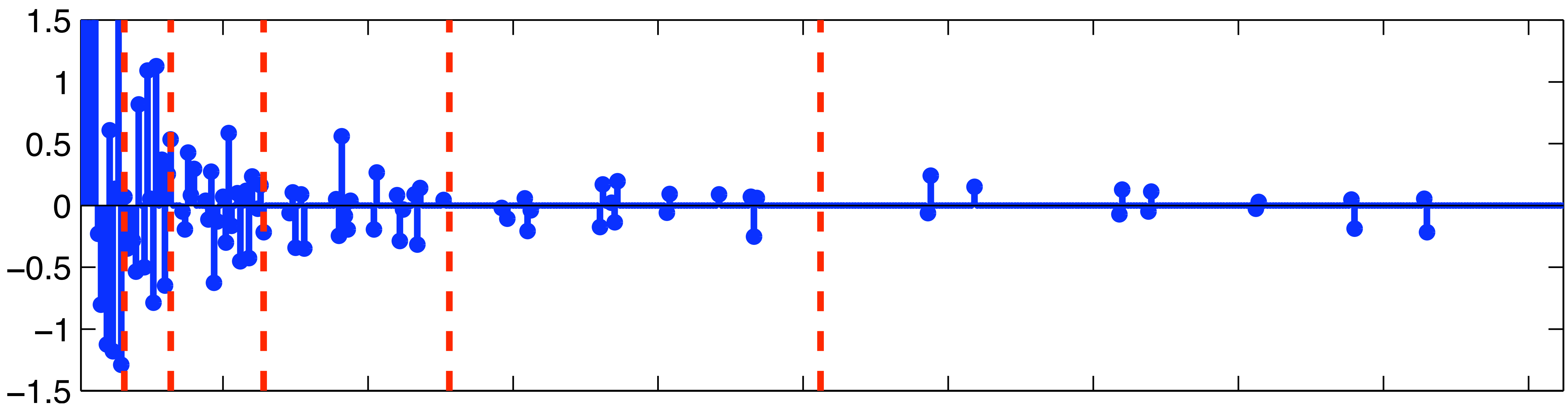

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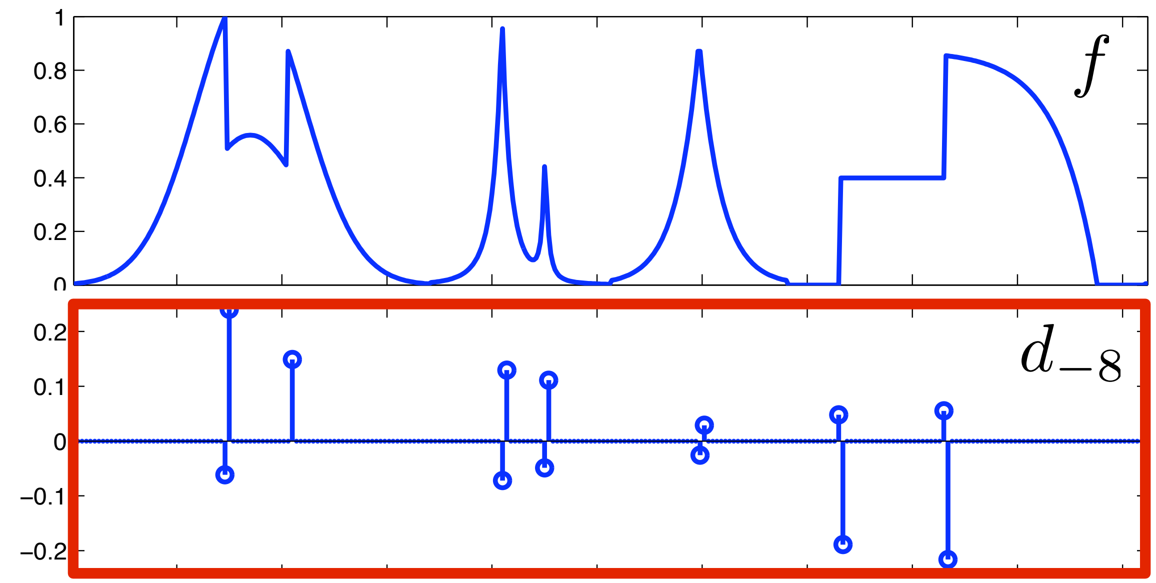

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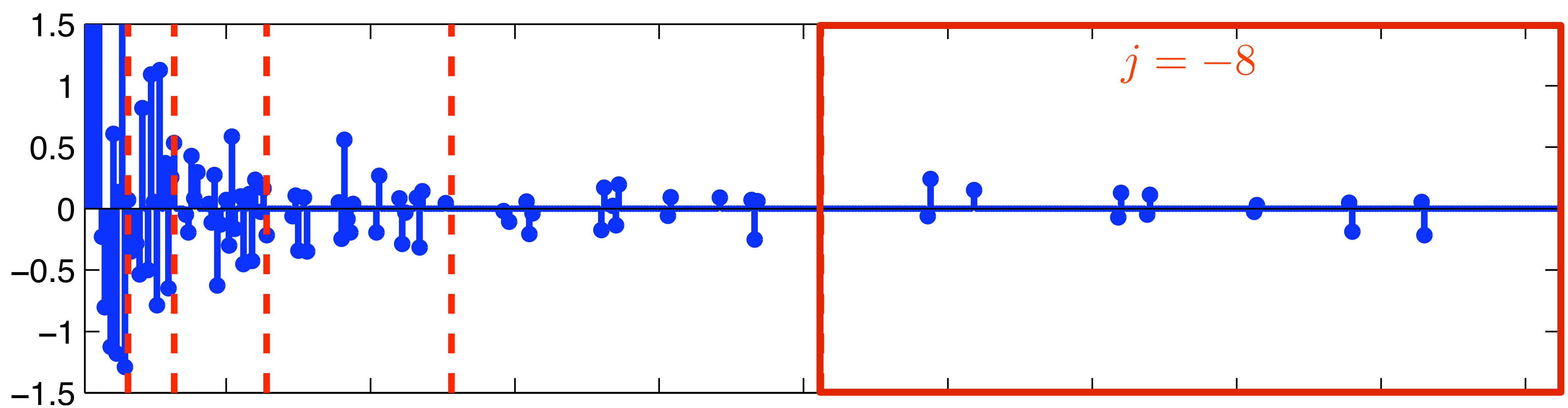

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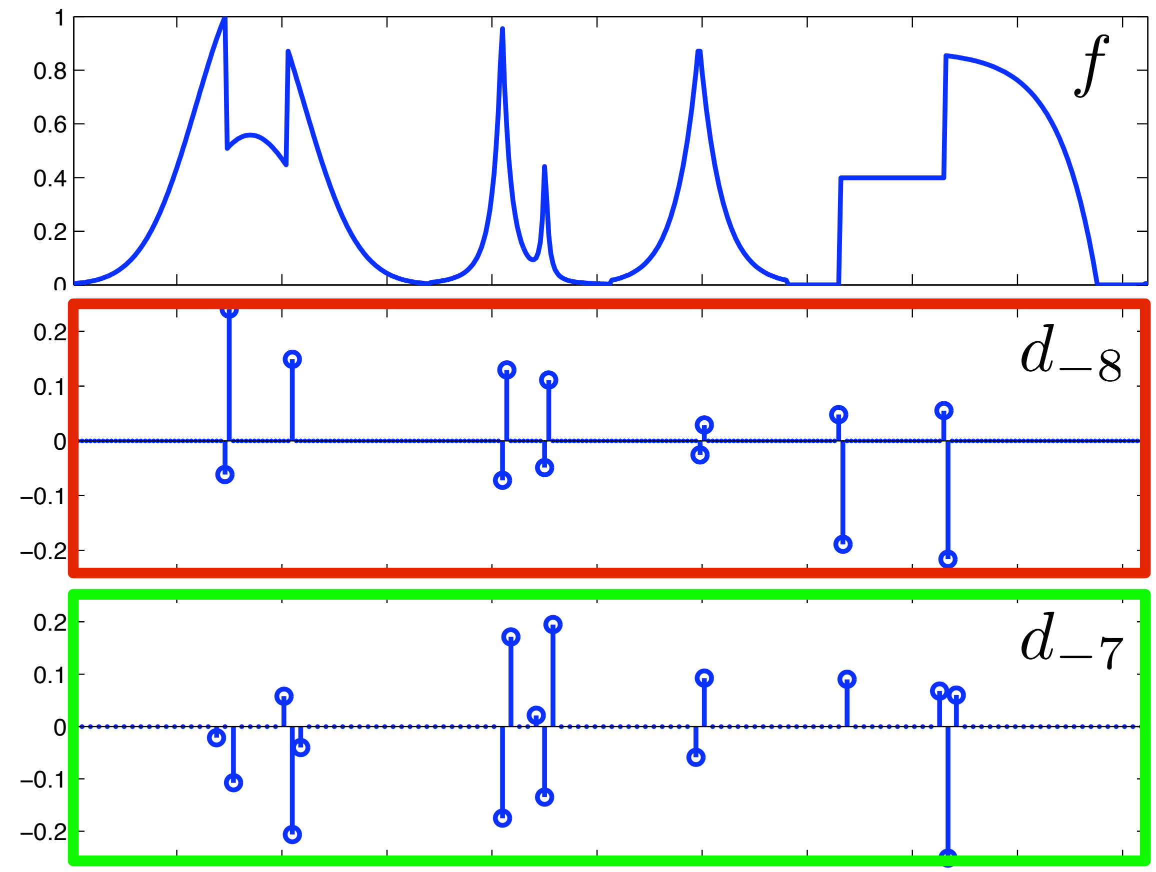

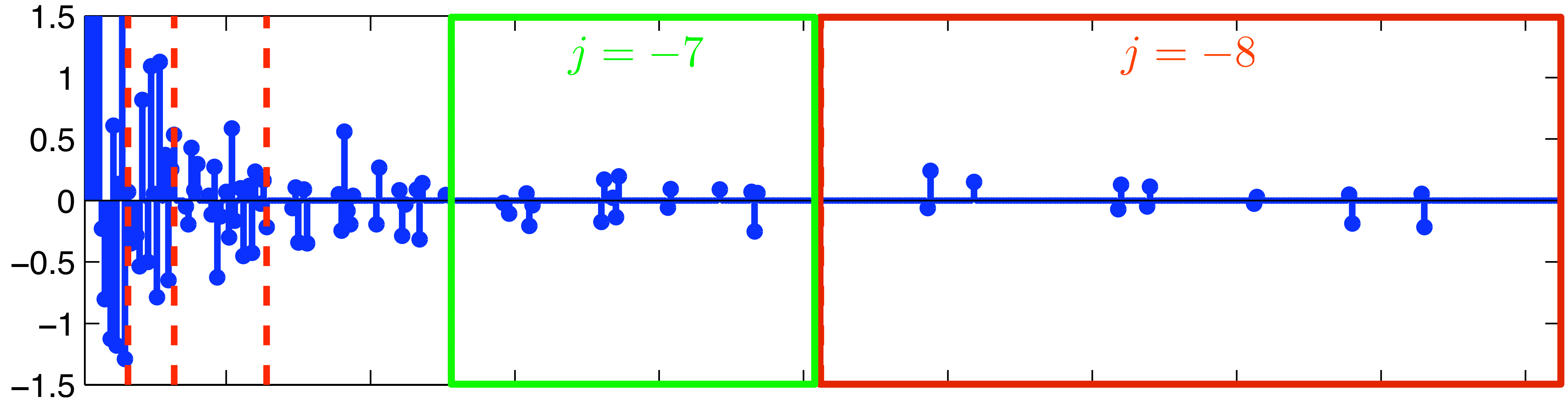

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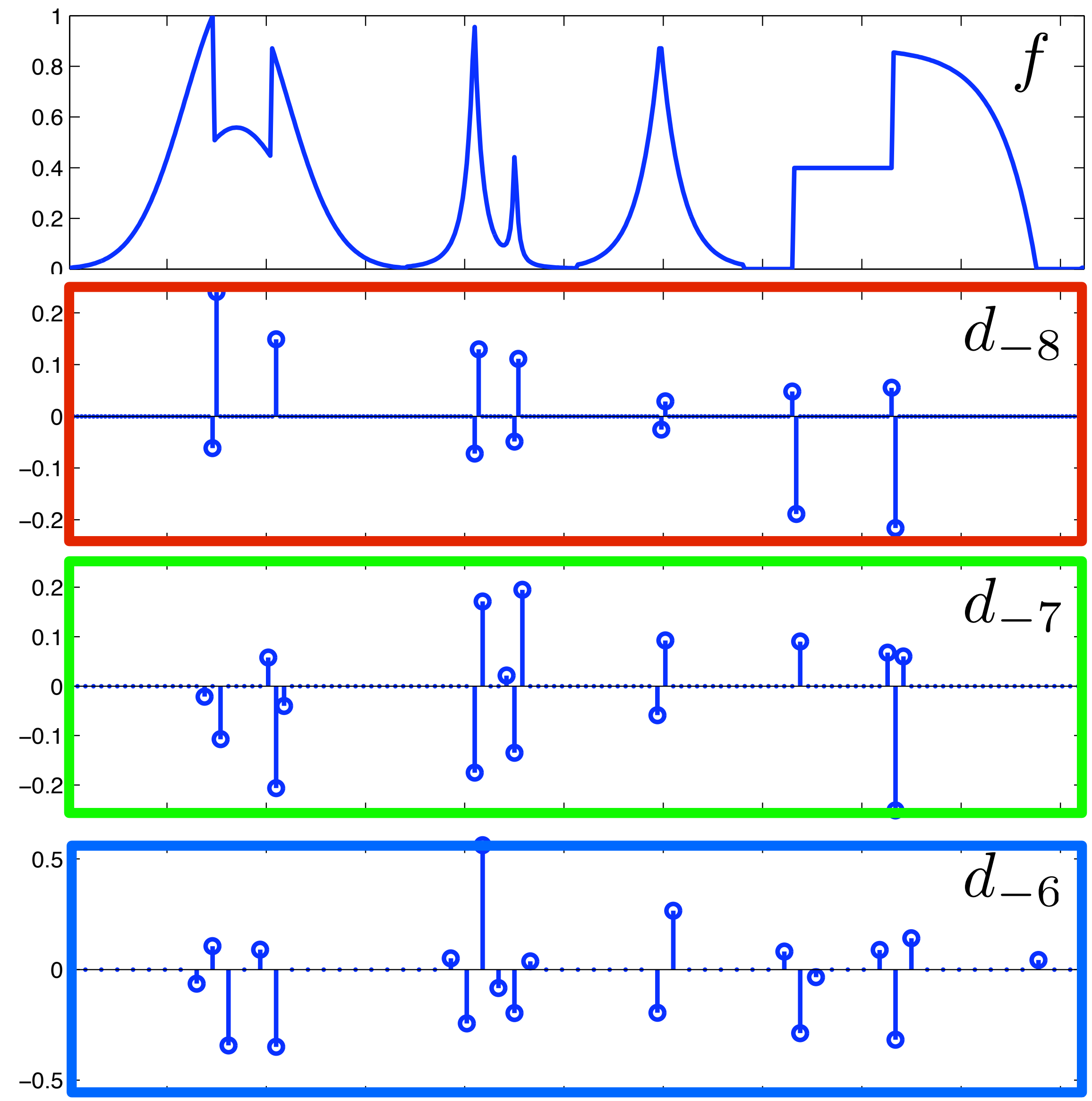

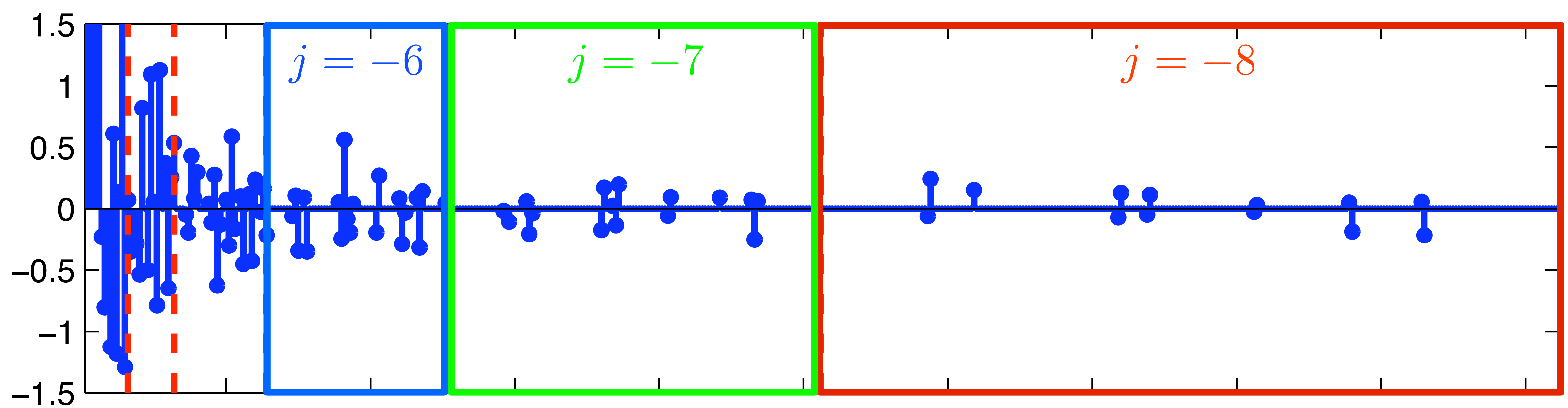

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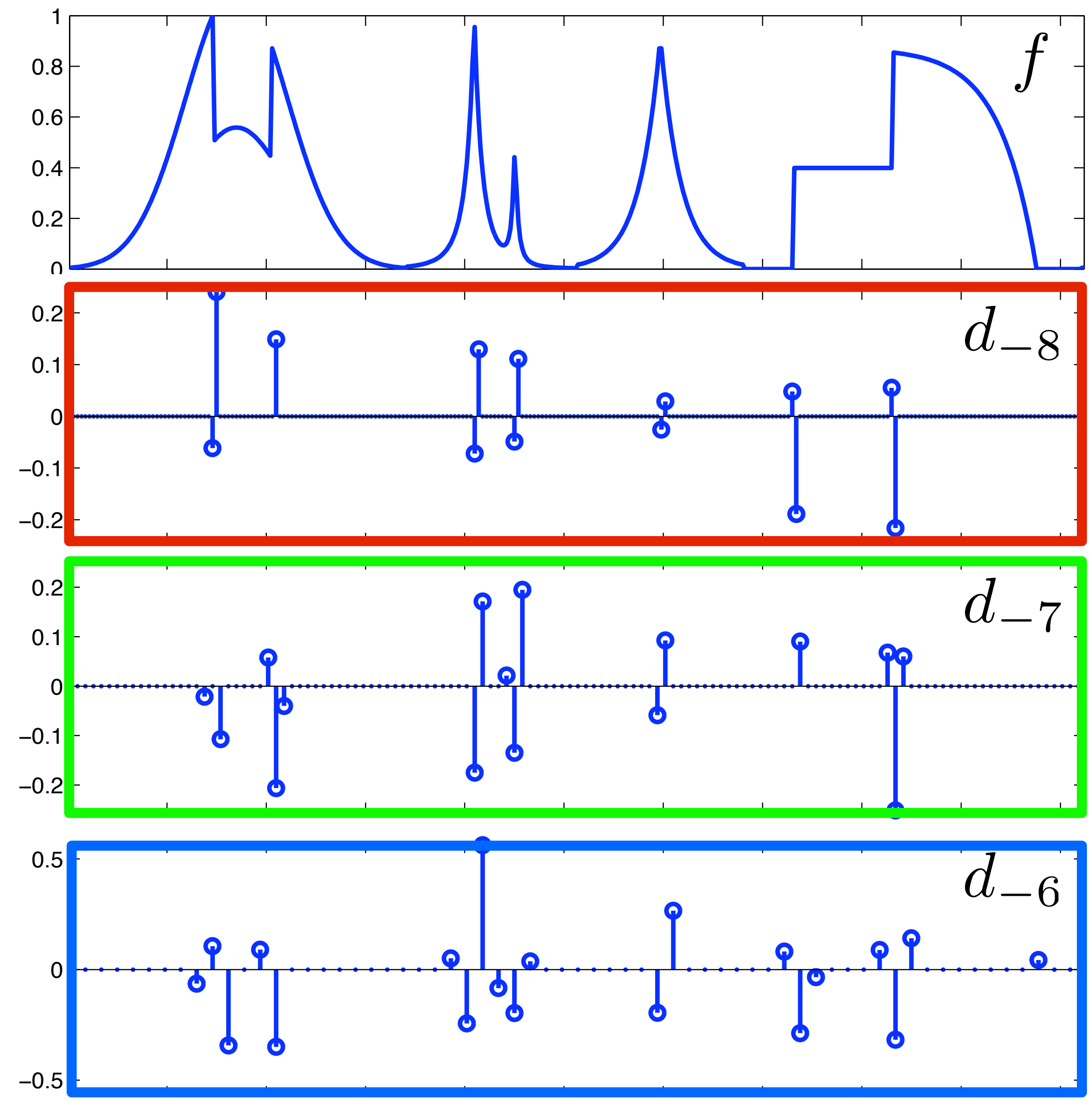

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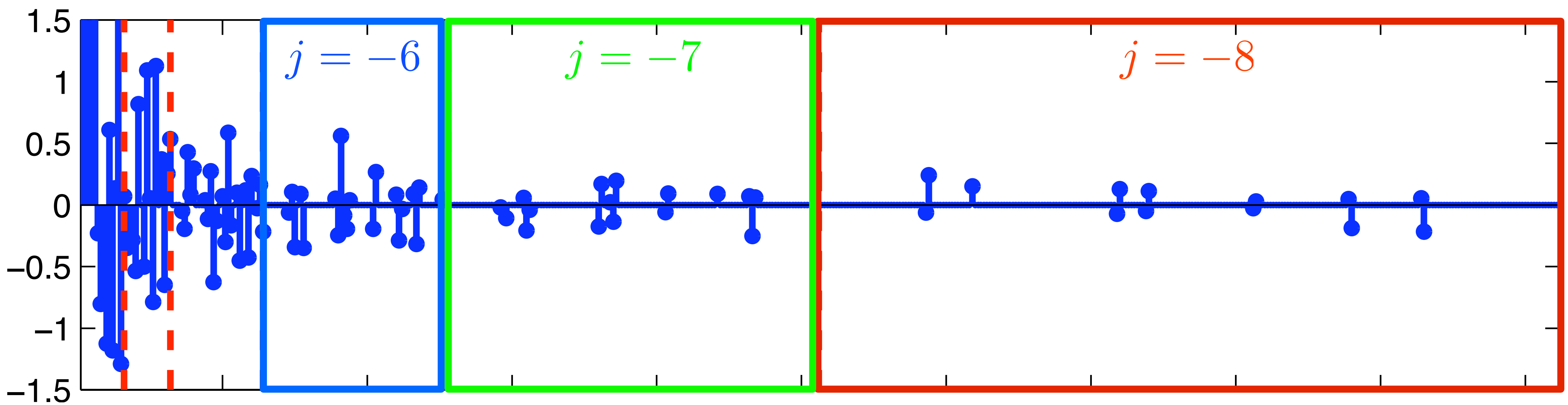
Discrete wavelet basis $\{\bar\psi_{j,n}\}_{j,n}$ of $\mathbb{C}^N$,

for $J < j \leqslant 0,\ 0 \leqslant n < 2^{-j}$.



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Detail coefficients: $d_j[n] = \langle f,\, \psi_{j,n} \rangle$.

Refinement relations:

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right) = \sum_{n\in\mathbb{Z}} h[n]\varphi(t-n) \qquad \frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n\in\mathbb{Z}} g[n]\varphi(t-n)$$

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$n \to n - 2p$

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$h \approx$ low pass filter. $g \approx$ high pass filter.

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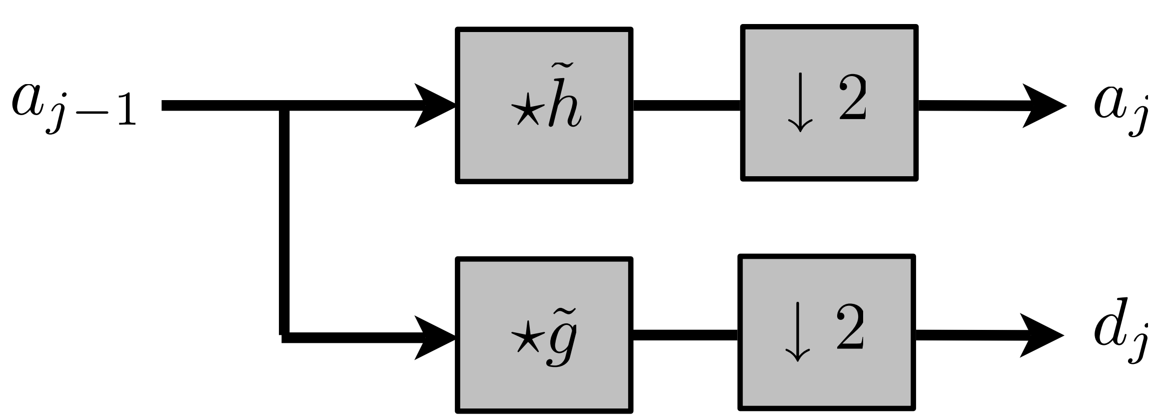
Computing (a_j, d_j) from a_{j-1}: $\tilde{x}[n] = x[-n]$

$$a_j[p] = \sum_{n\in\mathbb{Z}} h[n-2p]a_{j-1}[n] = a_{j-1} \star \tilde{h}[2p] \qquad d_j[p] = \sum_{n\in\mathbb{Z}} g[n-2p]a_{j-1}[n] = a_{j-1} \star \tilde{g}[2p]$$

$\longrightarrow$ low pass + sub-sampling $\longrightarrow$ high pass + sub-sampling



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$$\frac{1}{\sqrt{2}}\varphi\left(\frac{t-2^j p}{2^j}\right) = \sum_{n\in\mathbb{Z}} h[n]\varphi\left(\frac{t}{2^{j-1}} - (n+2p)\right)$$

$$n \to n - 2p$$

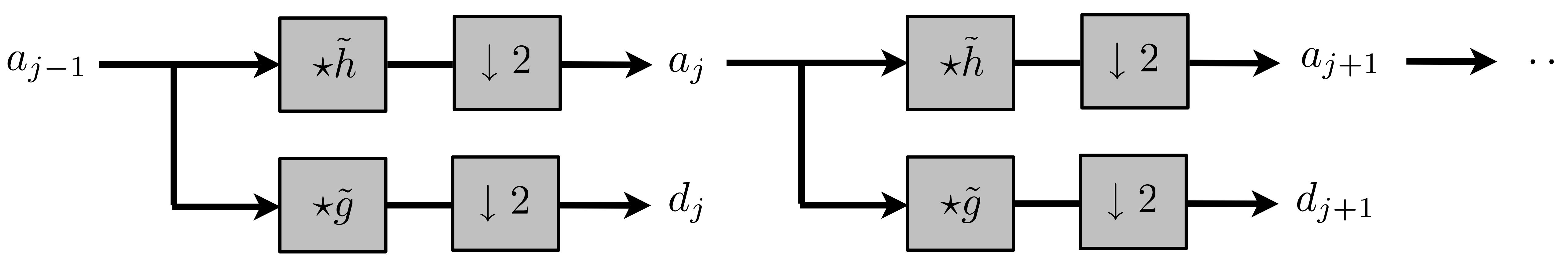
$$\varphi_{j,p} = \sum_{n\in\mathbb{Z}} h[n-2p]\varphi_{j-1,n} \qquad \psi_{j,p} = \sum_{n\in\mathbb{Z}} g[n-2p]\varphi_{j-1,n}$$

$h \approx$ low pass filter. $g \approx$ high pass filter.

Computing (a_j, d_j) from a_{j-1}: $\tilde{x}[n] = x[-n]$

$$a_j[p] = \sum_{n\in\mathbb{Z}} h[n-2p]a_{j-1}[n] = a_{j-1} \star \tilde{h}[2p] \qquad d_j[p] = \sum_{n\in\mathbb{Z}} g[n-2p]a_{j-1}[n] = a_{j-1} \star \tilde{g}[2p]$$

$\longrightarrow$ low pass + sub-sampling $\longrightarrow$ high pass + sub-sampling



Fast Wavelet Transform

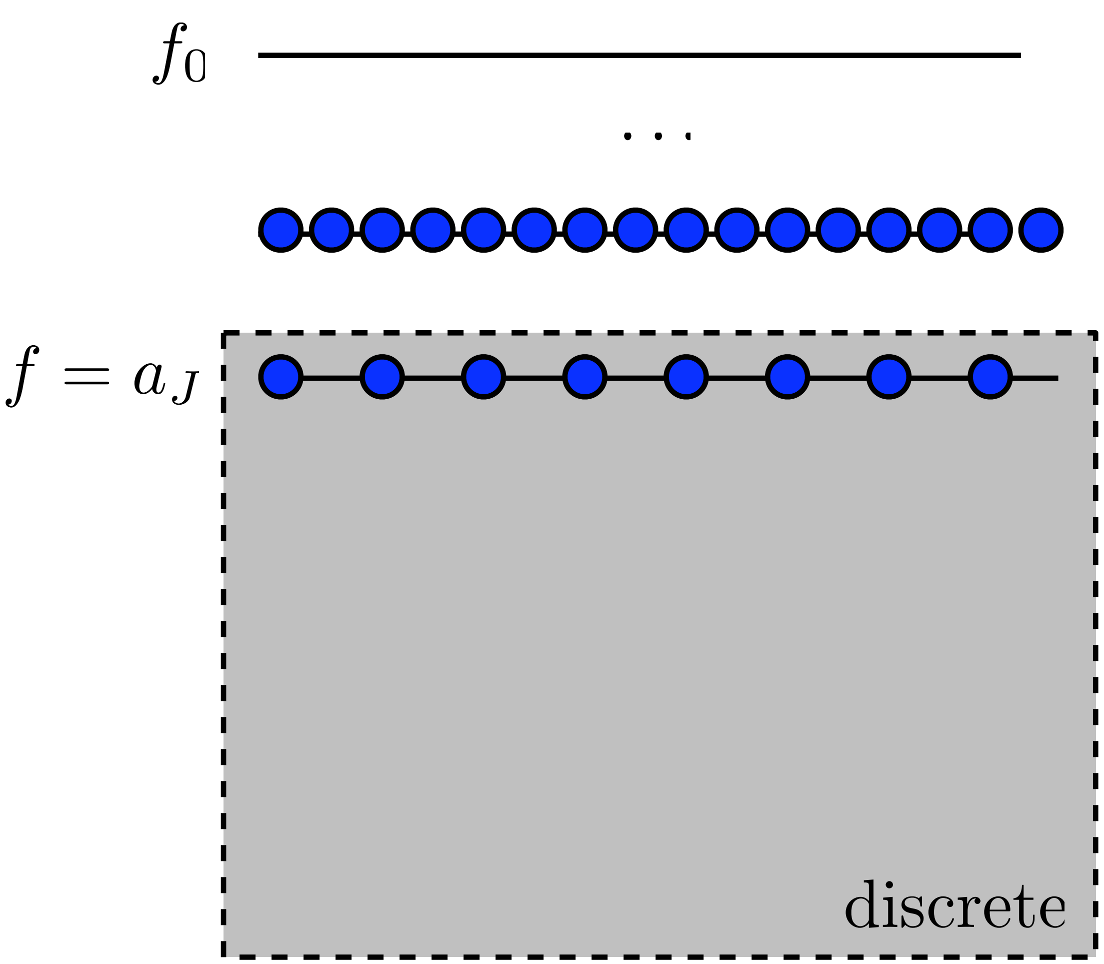
Fast wavelet transform:

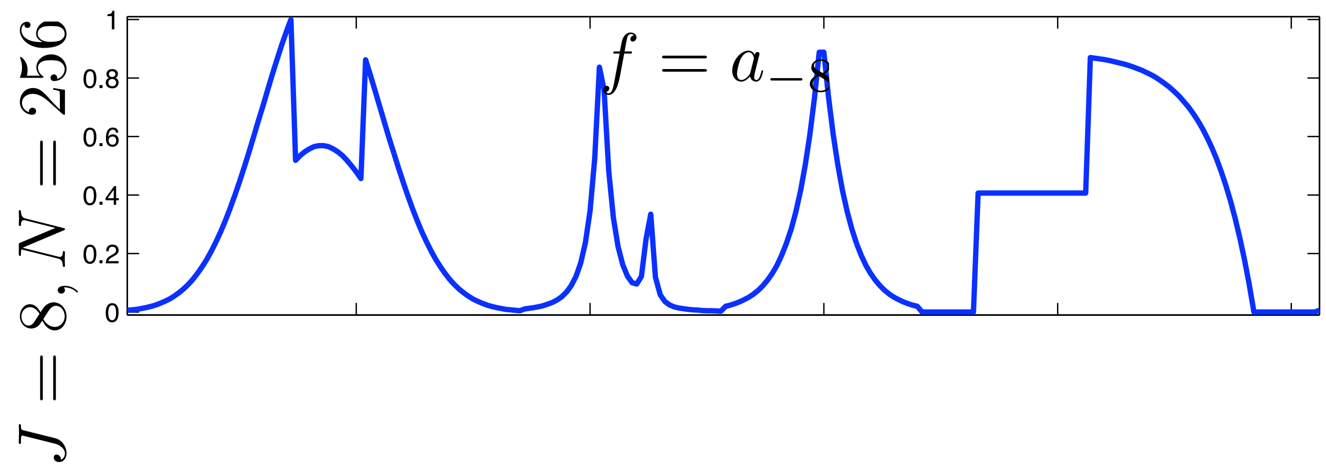
Initialization: $a_J = f$.

For $j = J, \ldots, 0$

$$a_{j+1} = (a_j \star \tilde{h}) \downarrow 2$$

$$d_{j+1} = (a_j \star \tilde{g}) \downarrow 2$$





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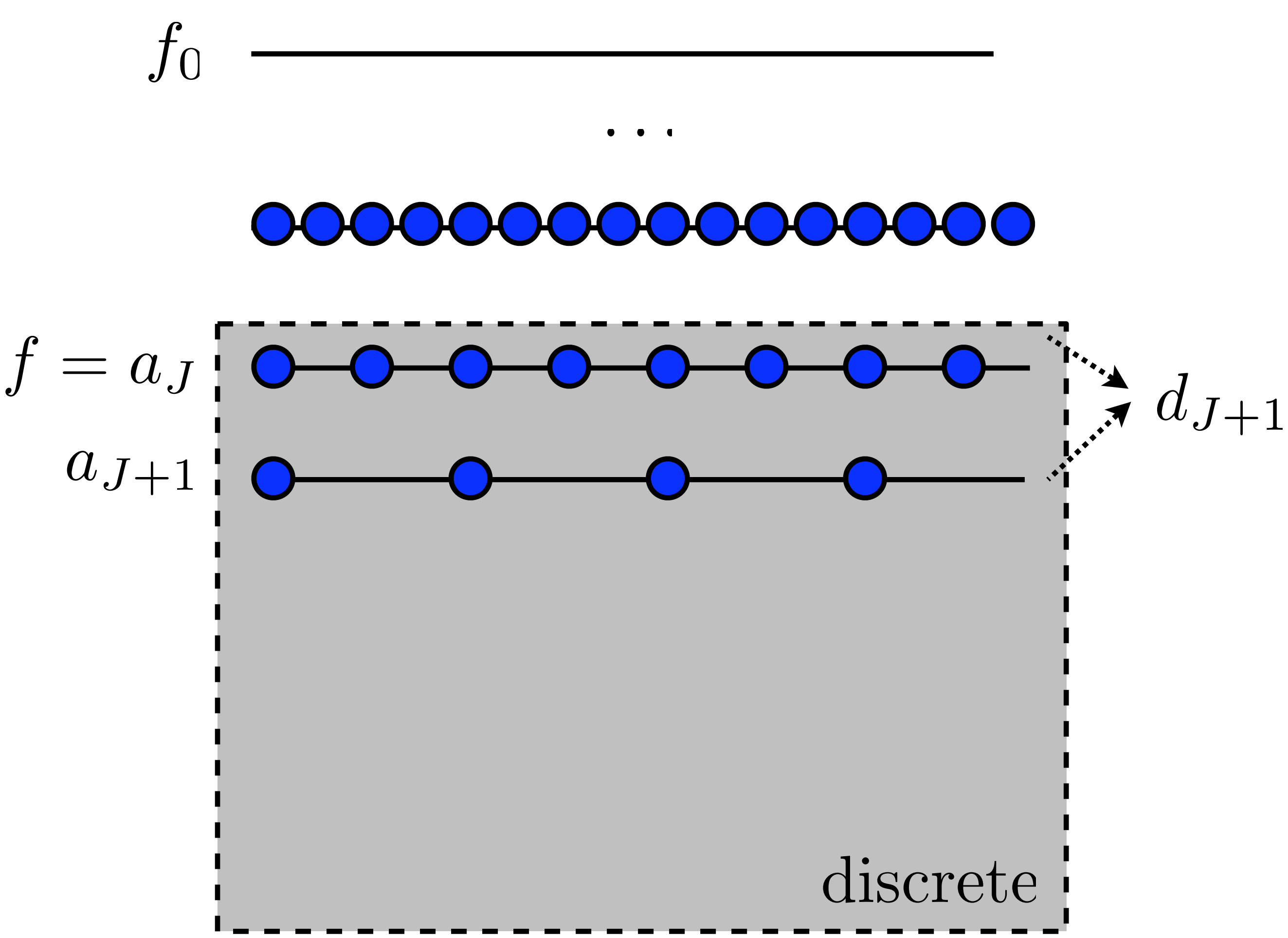

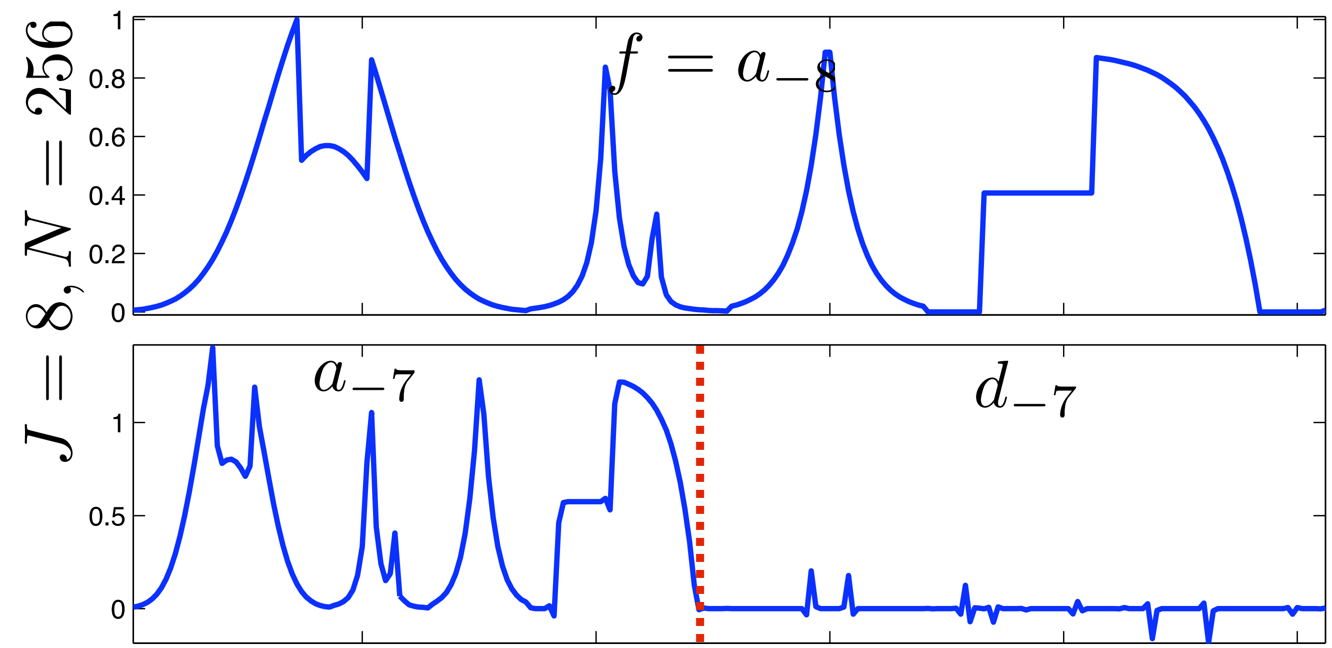

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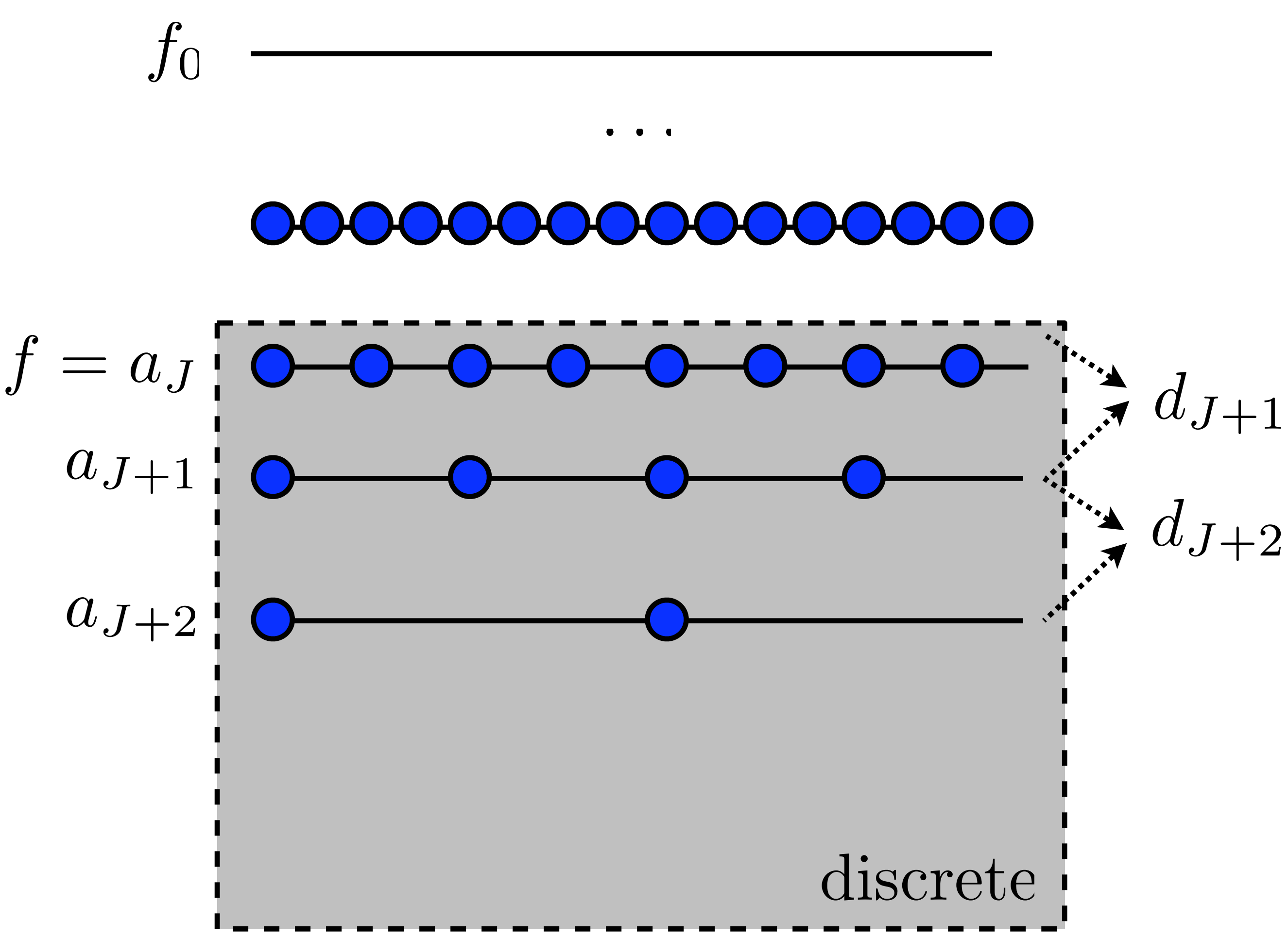

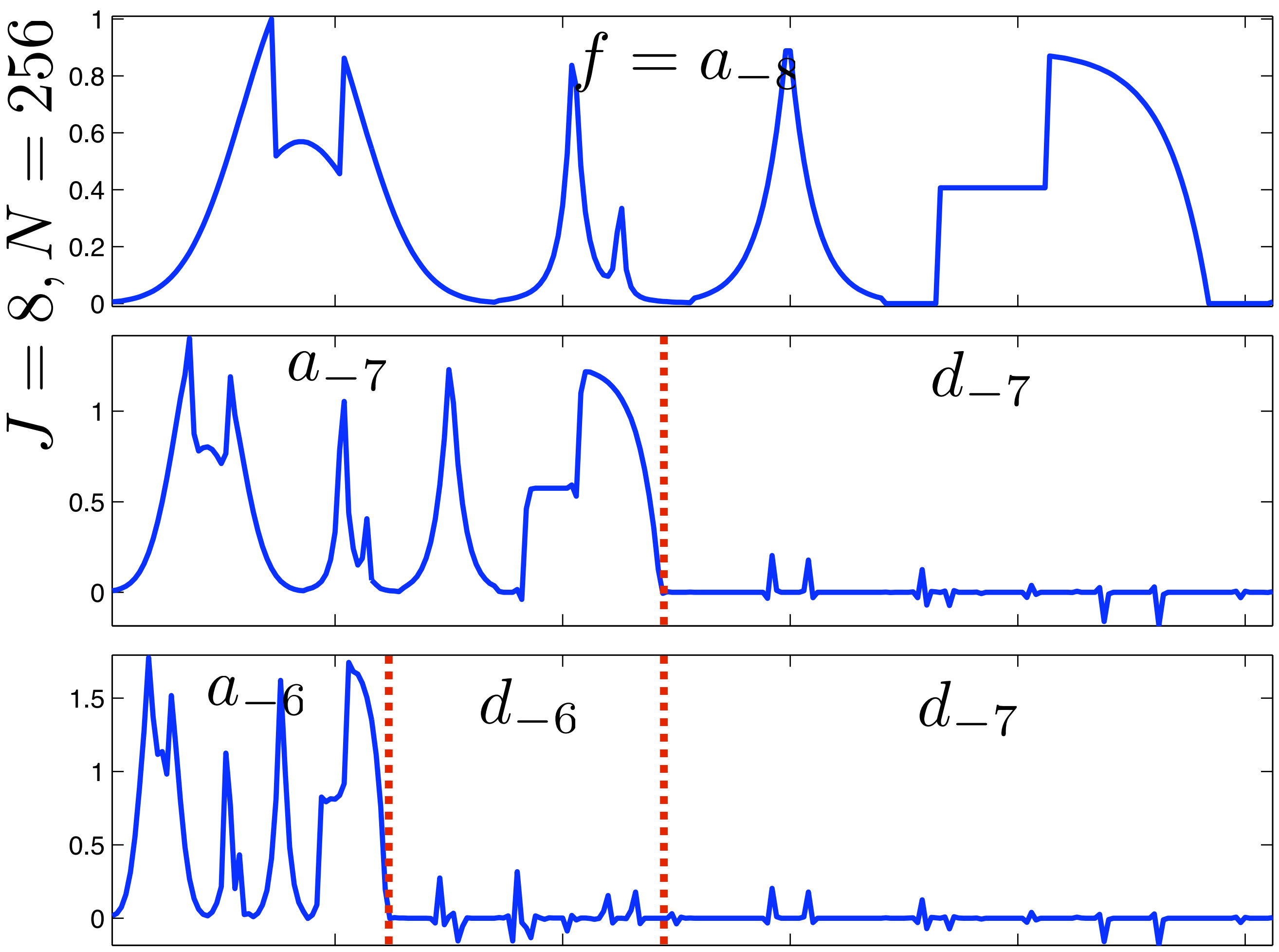

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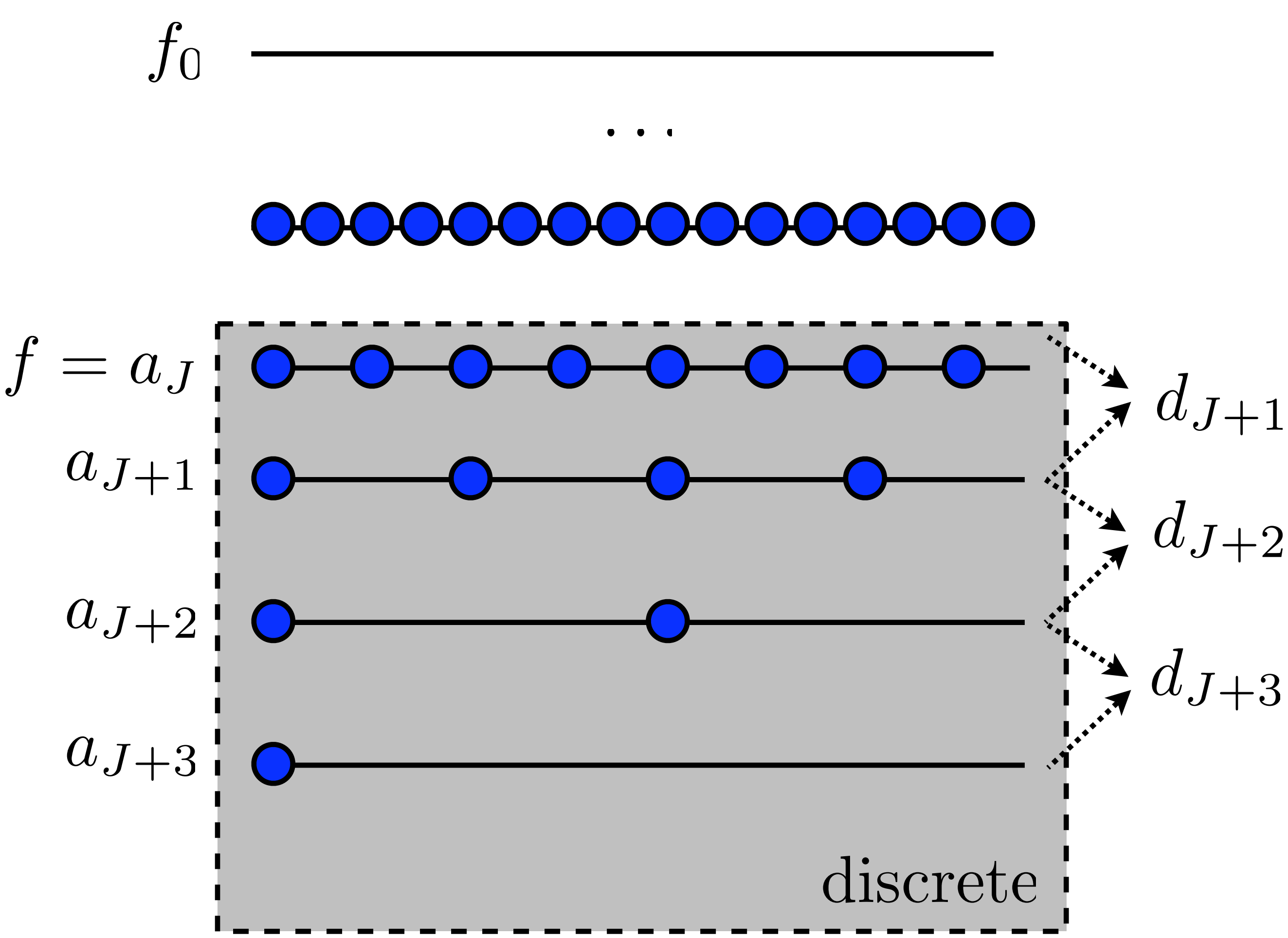

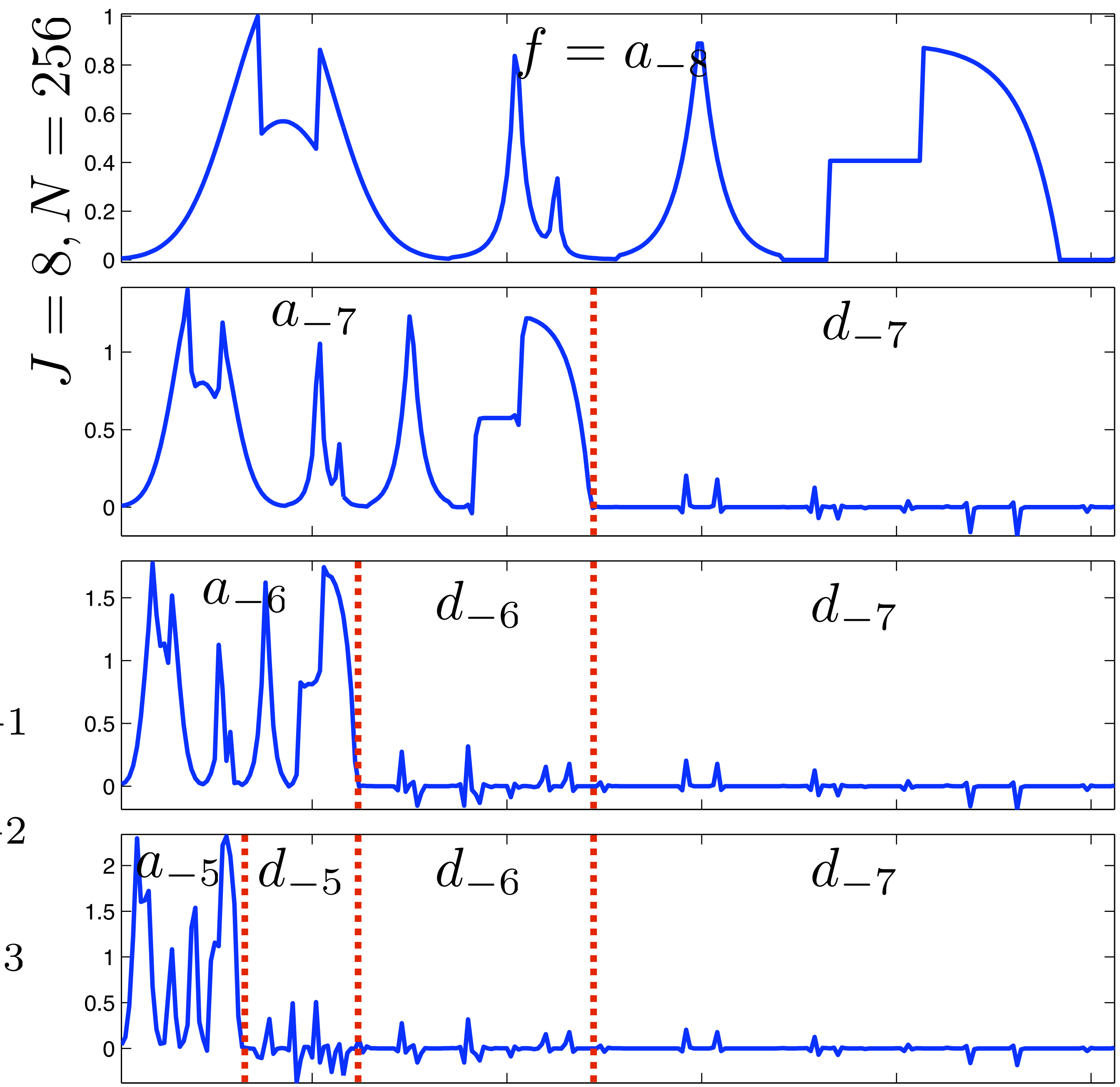

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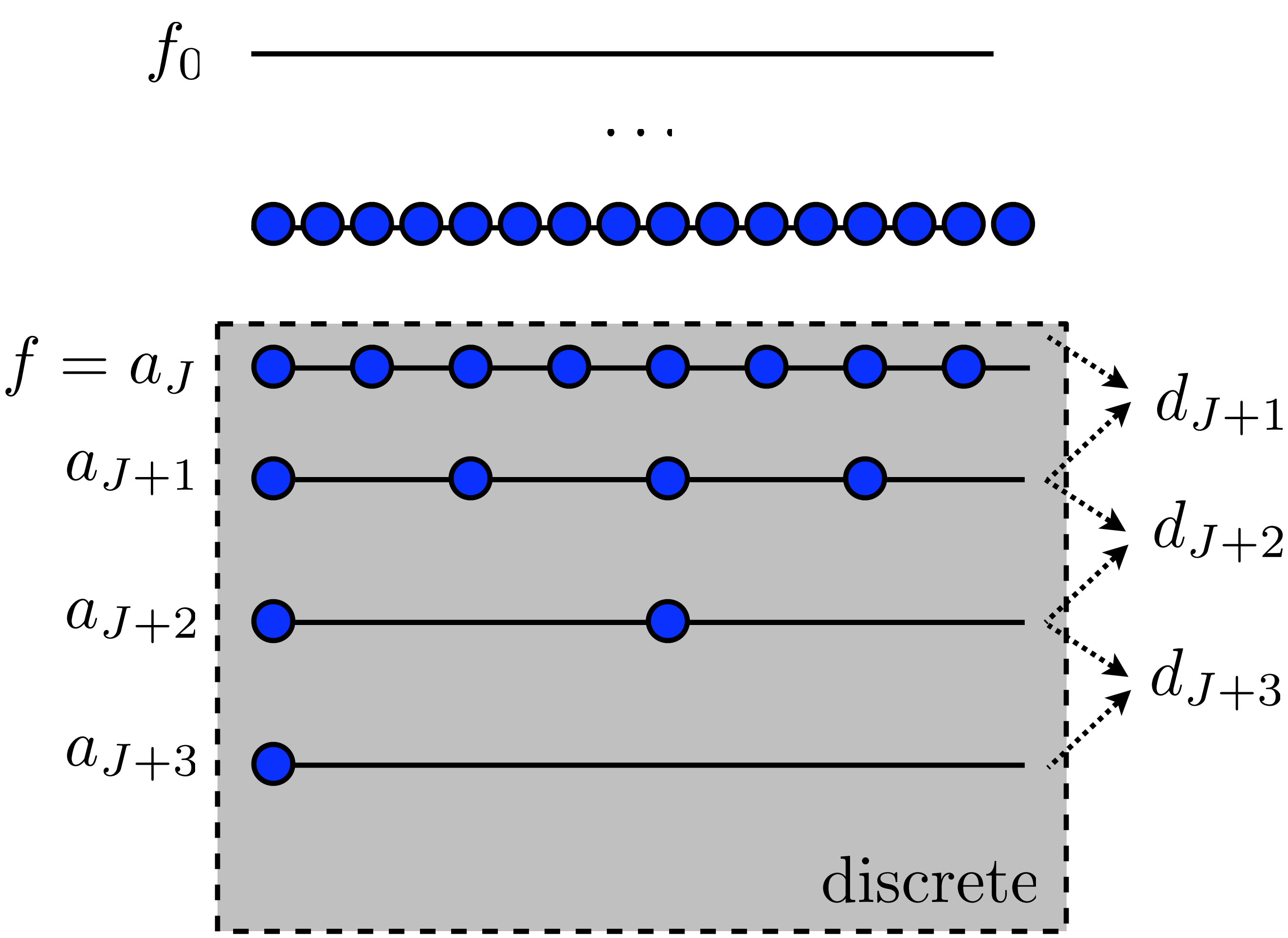
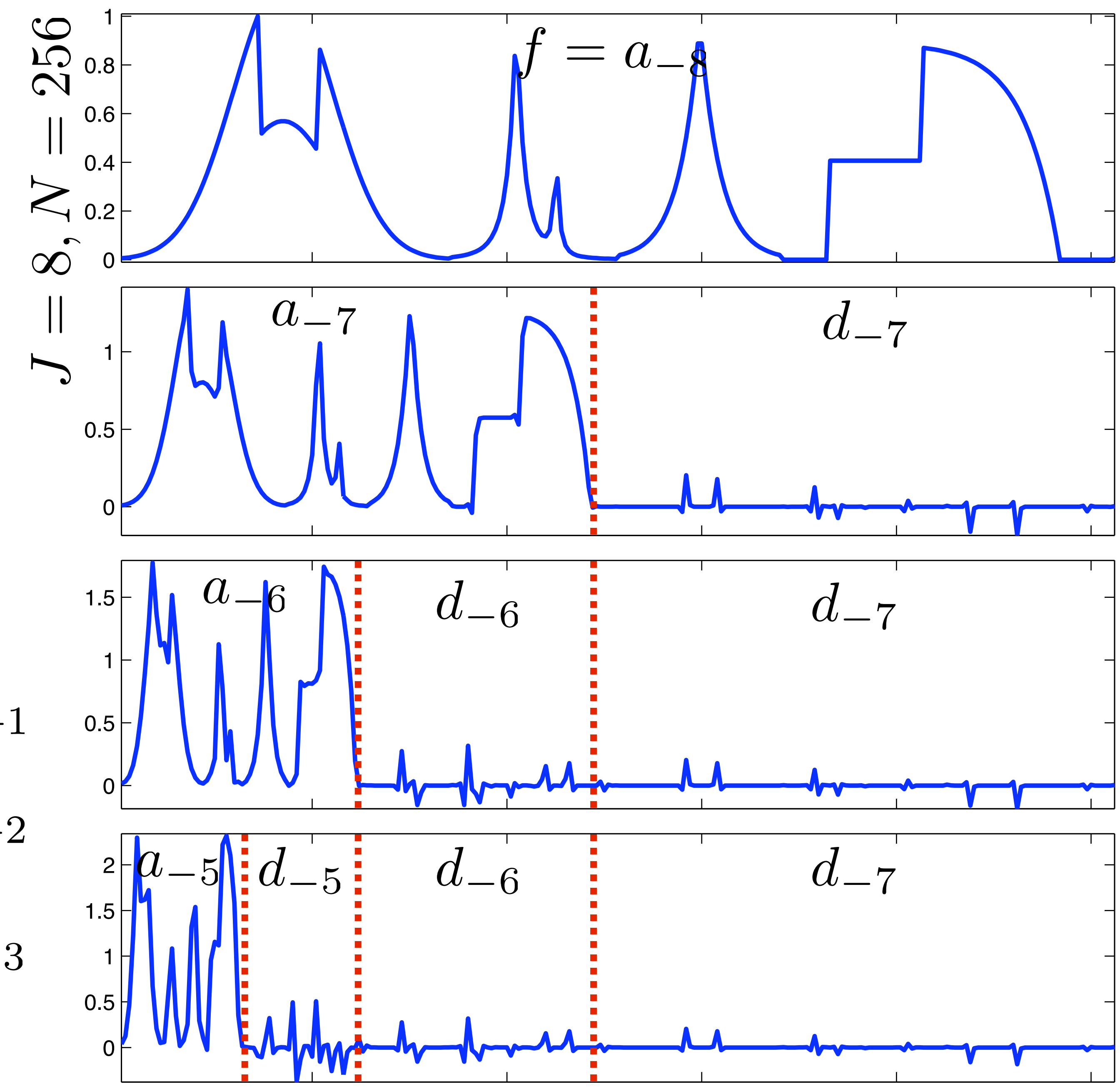
Fast wavelet transform:

$$\boxed{\begin{array}{l}\text{Initialization: } a_J = f.\\ \text{For } j = J, \ldots, 0\\ \quad a_{j+1} = (a_j \star \tilde{h}) \downarrow 2\\ \quad d_{j+1} = (a_j \star \tilde{g}) \downarrow 2\end{array}}$$

Complexity: $$\sum_{j=-\log_2(N)}^{0} 2^j N(|h| + |g|)$$

$$\Longrightarrow \quad O(N \times (|h| + |g|))$$





Haar Refinement

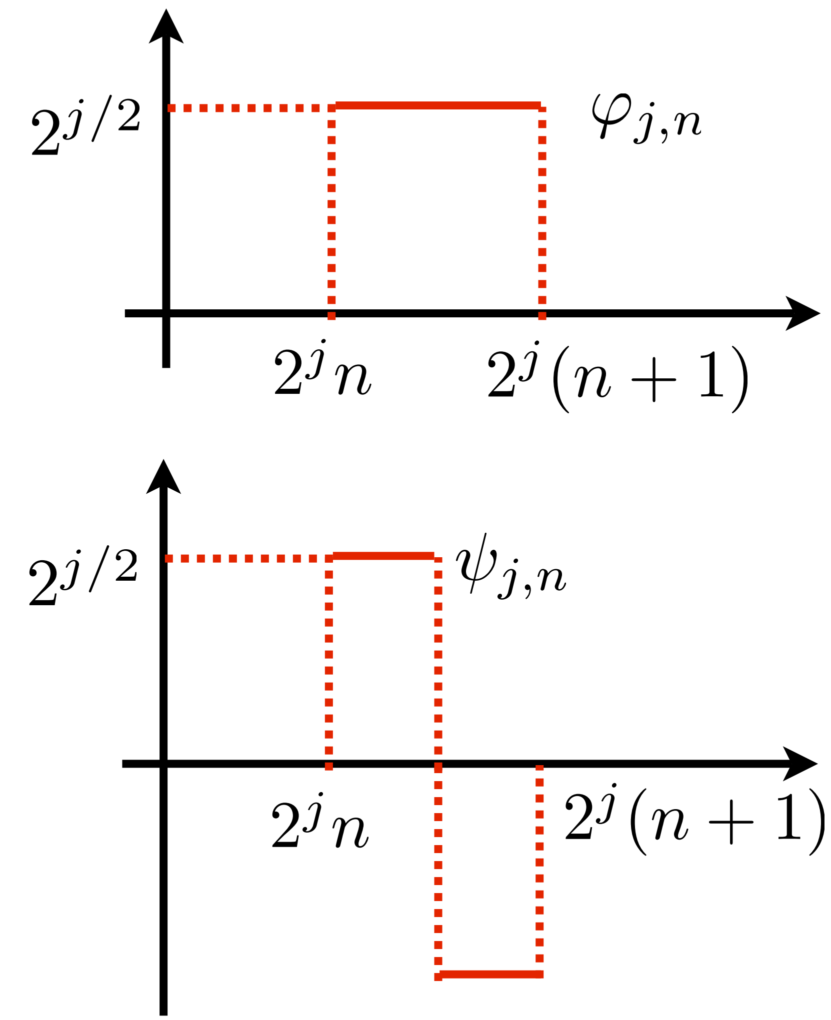
Refinement equations:

$$\varphi_{j,n} = \frac{1}{\sqrt{2}}(\varphi_{j-1,2n} + \varphi_{j-1,2n+1})$$

$$\psi_{j,n} = \frac{1}{\sqrt{2}}(\varphi_{j-1,2n} - \varphi_{j-1,2n+1})$$

Low-pass: $h = [\ldots,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ldots]$

High-pass: $g = [\ldots,\ 0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0,\ldots]$



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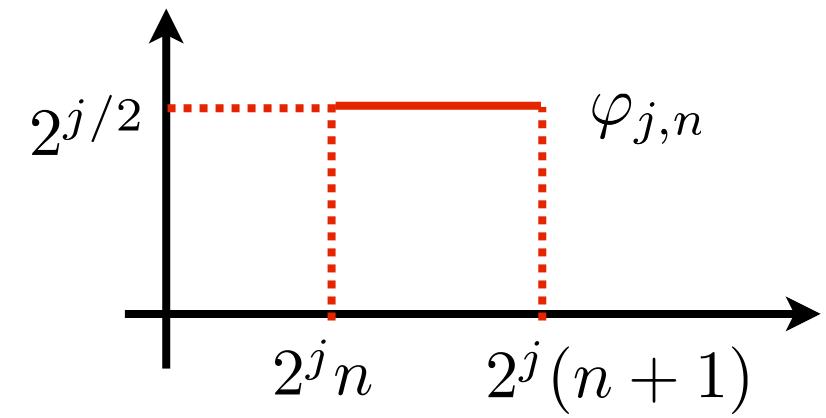
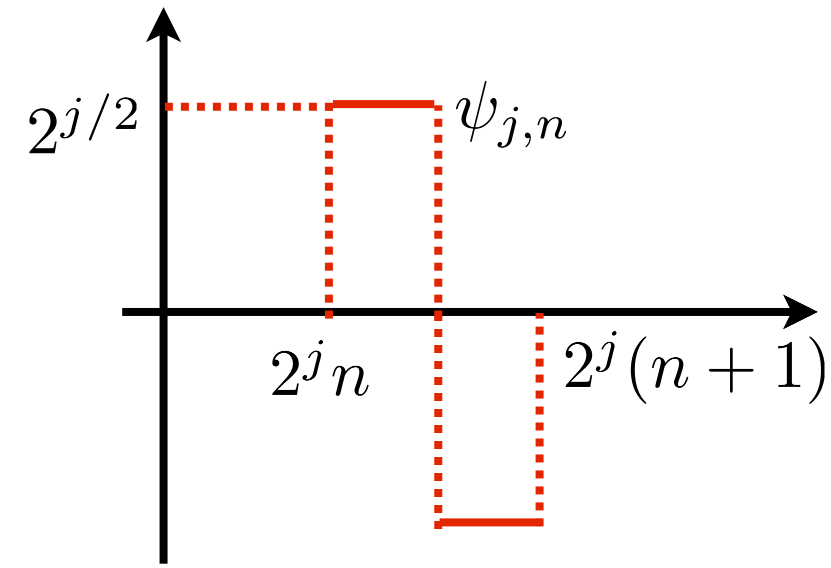




Haar transform algorithm:

Initialization: $a_J = f$.

For $j = J, \ldots, 1$

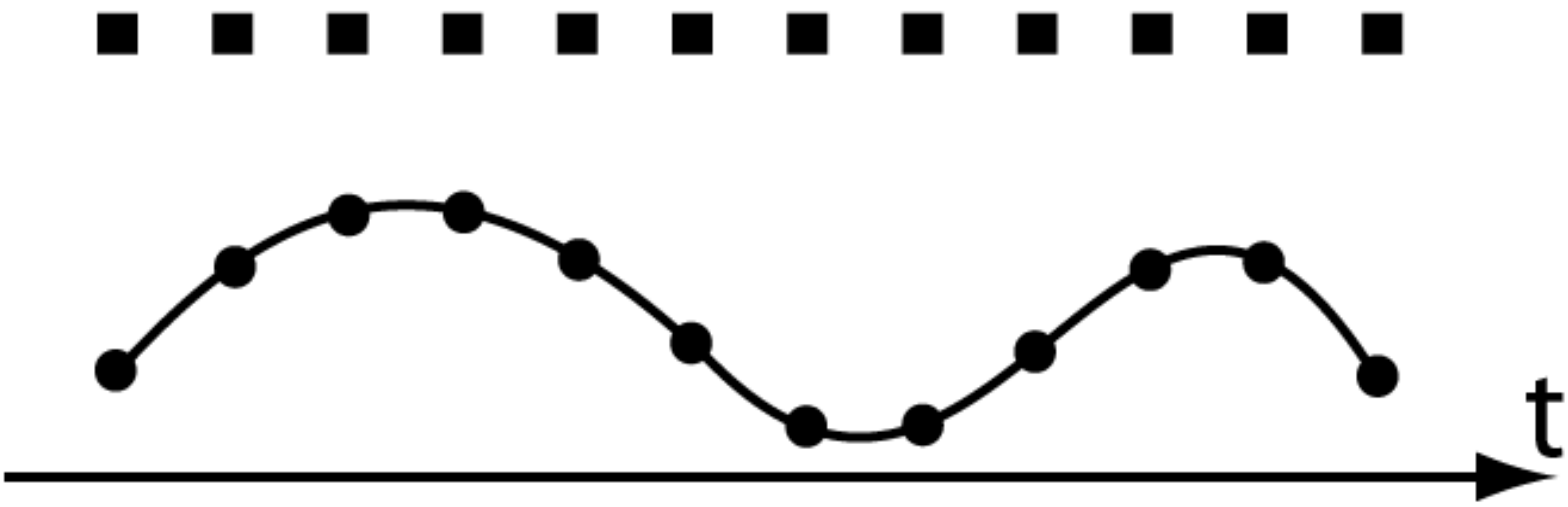
$$a_j[n] = \frac{1}{\sqrt{2}}(a_{j-1}[2n] + a_{j-1}[2n+1])$$

$$d_j[n] = \frac{1}{\sqrt{2}}(a_{j-1}[2n] - a_{j-1}[2n+1])$$

Haar Transform

Elementary step: $(x, y) \longmapsto (a, d) = (x + y, x - y)/\sqrt{2}$.

Inverse step: $(a, d) \longmapsto (x, y) = (a + d, a - d)/\sqrt{2}$.



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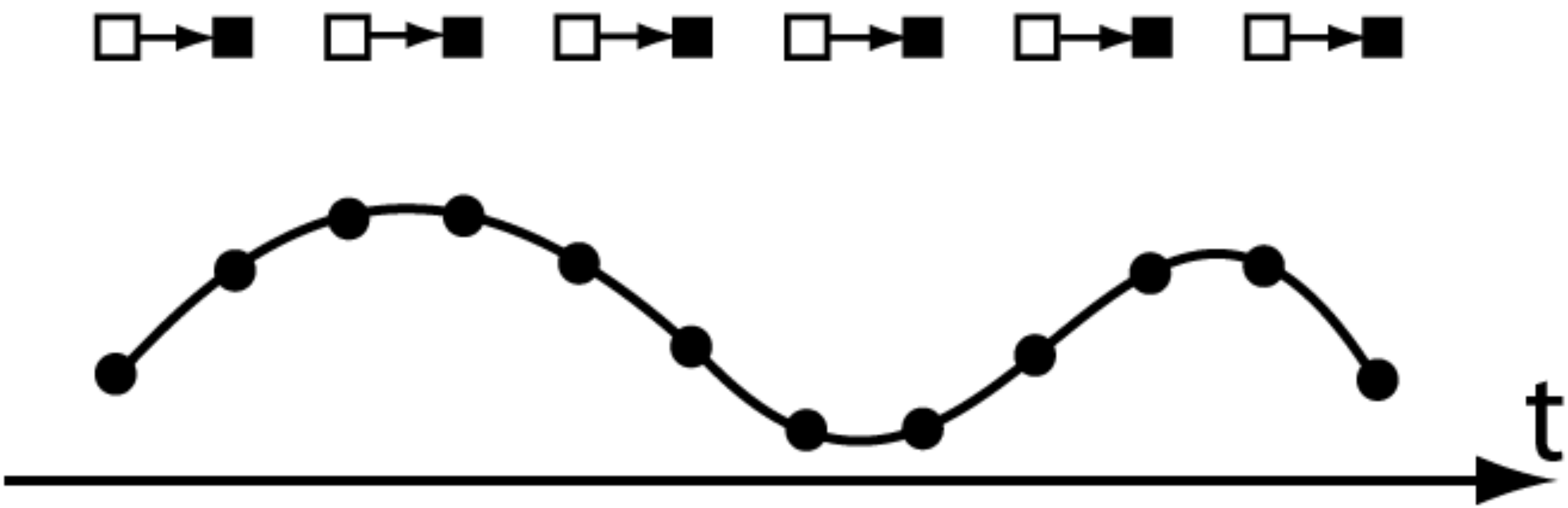

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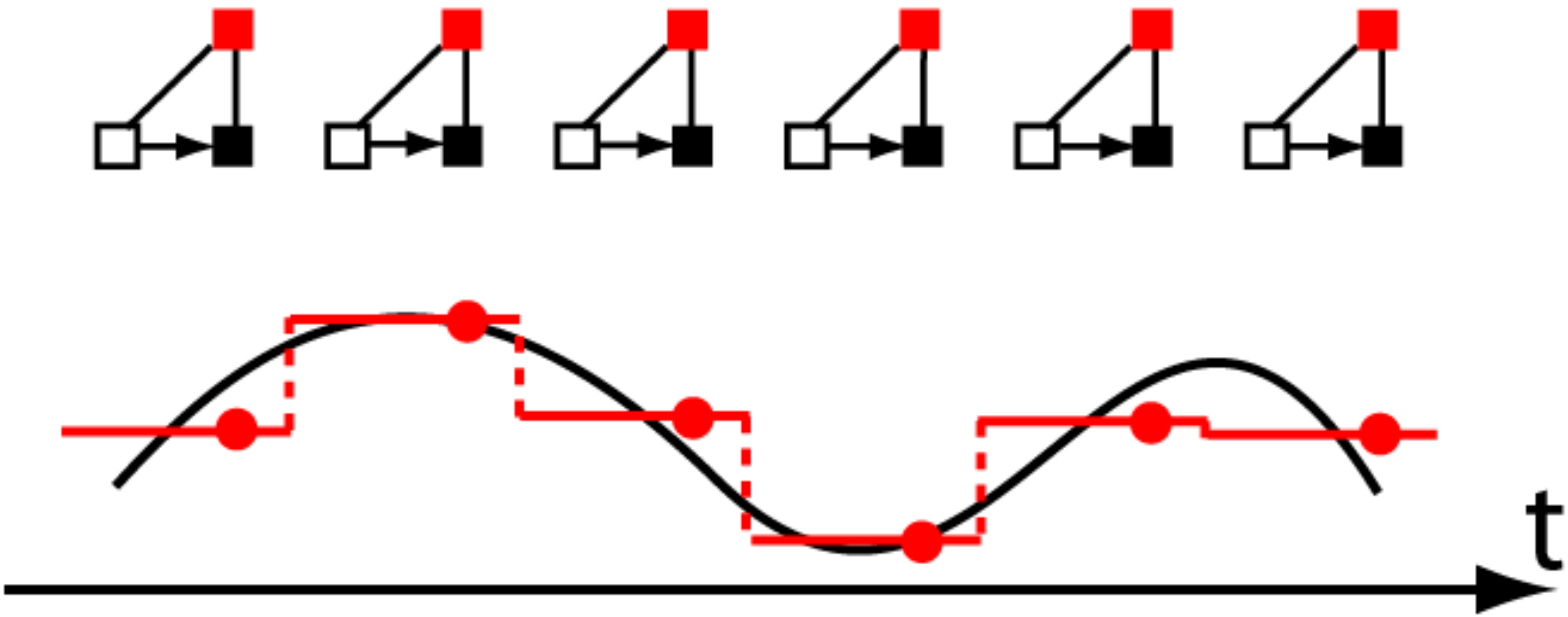

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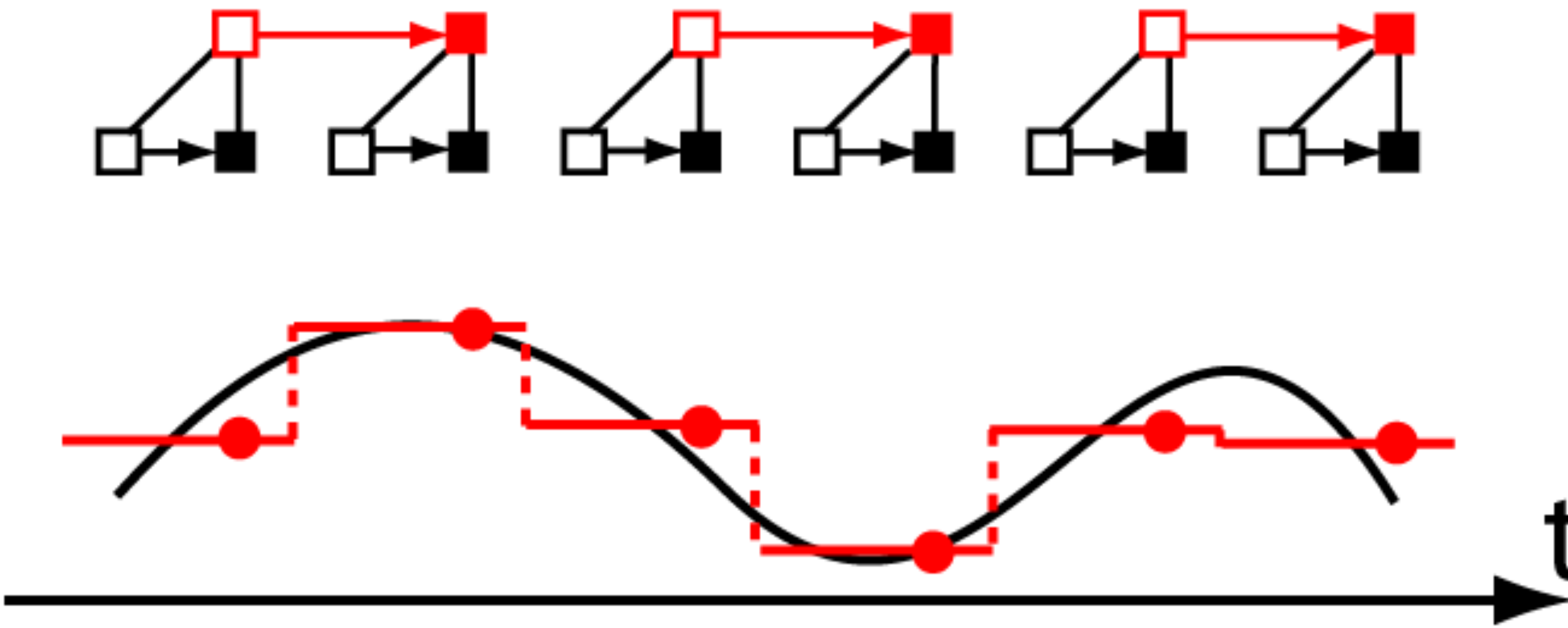

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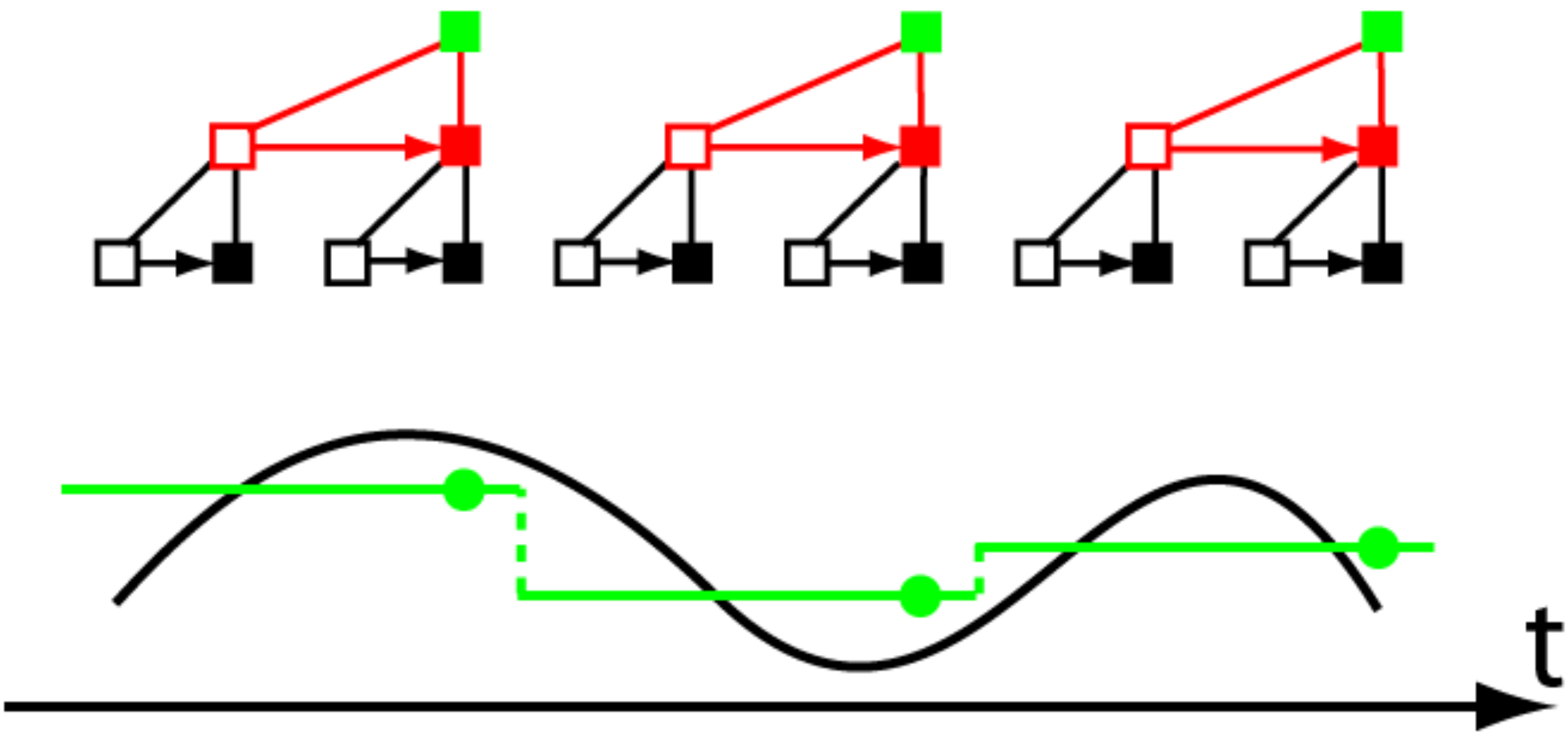

Inverting the Transform

Forward elementary step: $a_{j-1} \in \mathbb{R}^{2^{1-j}} \longmapsto (a_j, d_j) \in \mathbb{R}^{2^{-j}} \times \mathbb{R}^{2^{-j}}$.

Orthogonal mapping: $\|a_{j-1}\|^2 = \|a_j\|^2 + \|d_j\|^2$.

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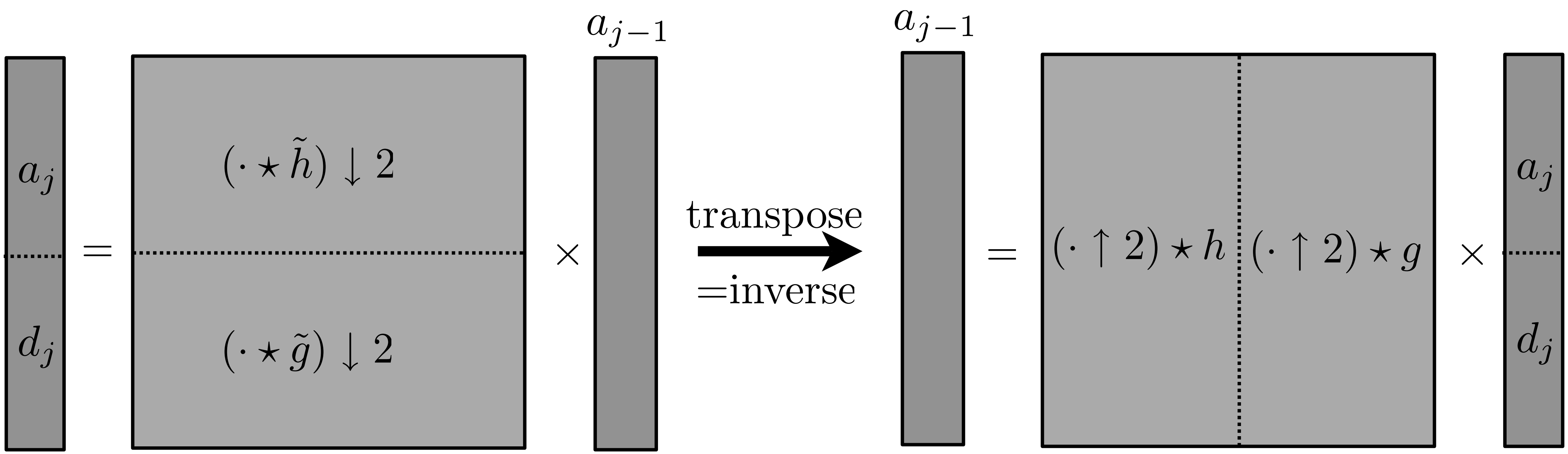

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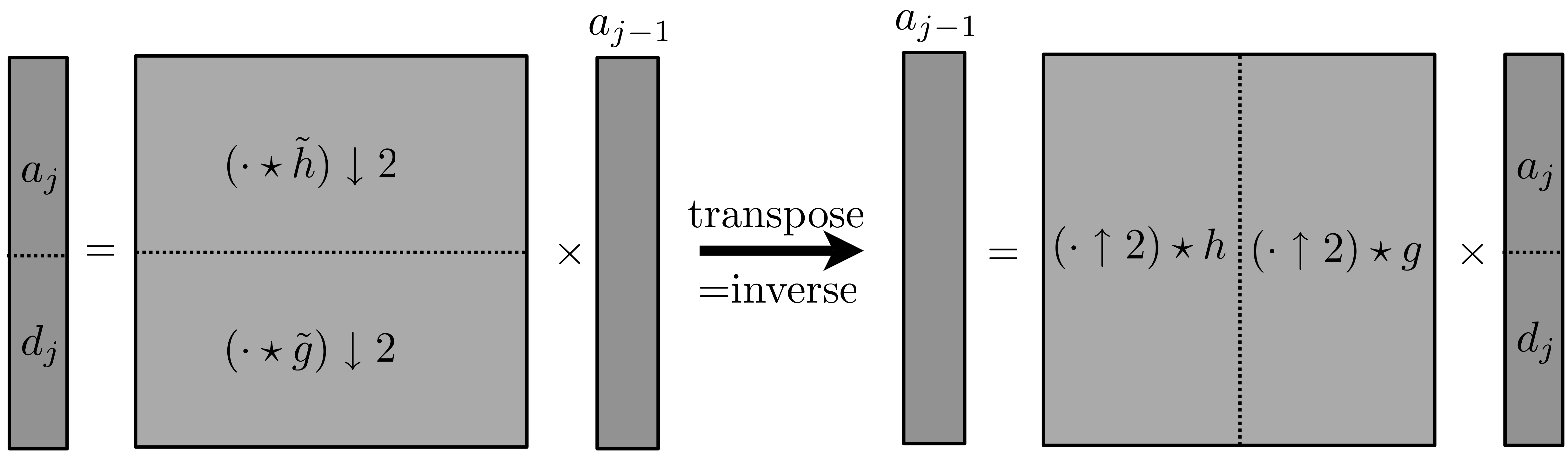
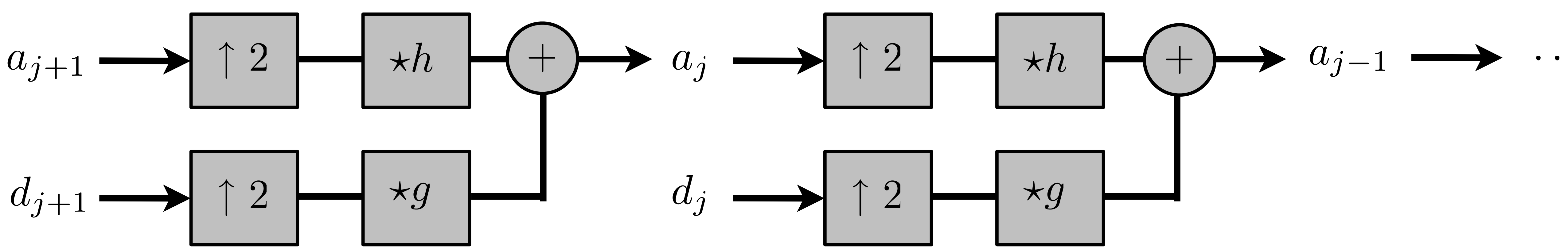


$$a_{j-1} = (a_j \uparrow 2) \star h + (d_j \uparrow 2) \star g$$



Overview

- Multiresolutions
- Fast Wavelet Transform
- **Wavelet Design**
- 2D Wavelet Processing

Approximation Filter Constraints

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right) = \sum_{n\in\mathbb{Z}} h[n]\varphi(t-n) \quad \Longrightarrow \quad \hat{\varphi}(2\omega) = \frac{1}{\sqrt{2}}\hat{h}[\omega]\hat{\varphi}(\omega)$$

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$$\Longrightarrow \quad \boxed{|\hat{h}[\omega]|^2 + |\hat{h}[\omega+\pi]|^2 = 2 \qquad (\mathbf{C}_2)}$$

$$\boxed{\textit{Theorem:} \quad \{\varphi_{j,n}\}_n \text{ ortho-basis of } V_j \quad \Longrightarrow \quad (\mathbf{C}_1) + (\mathbf{C}_2)}$$

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$$\boxed{\begin{array}{l} \hat h \text{ is } C^1 \text{ around } 0 \text{ and} \\ \displaystyle\inf_{\omega\in[-\pi/2,\pi/2]} |\hat h(\omega)| > 0 \end{array} \qquad (\mathbf{C}_3)}$$

Theorem: $\{\varphi_{j,n}\}_n$ ortho-basis of $V_j \Longrightarrow (\mathbf{C}_1) + (\mathbf{C}_2)$

Theorem: $(\mathbf{C}_1) + (\mathbf{C}_2) + (\mathbf{C}_3) \Longrightarrow \{\varphi_{j,n}\}_n$ ortho-basis of V_j

Detail Filter Constraint

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n\in\mathbb{Z}} g[n]\varphi(t-n) \quad \Longrightarrow \quad \hat{\psi}(2\omega) = \frac{1}{\sqrt{2}}\hat{g}[\omega]\hat{\varphi}(\omega)$$

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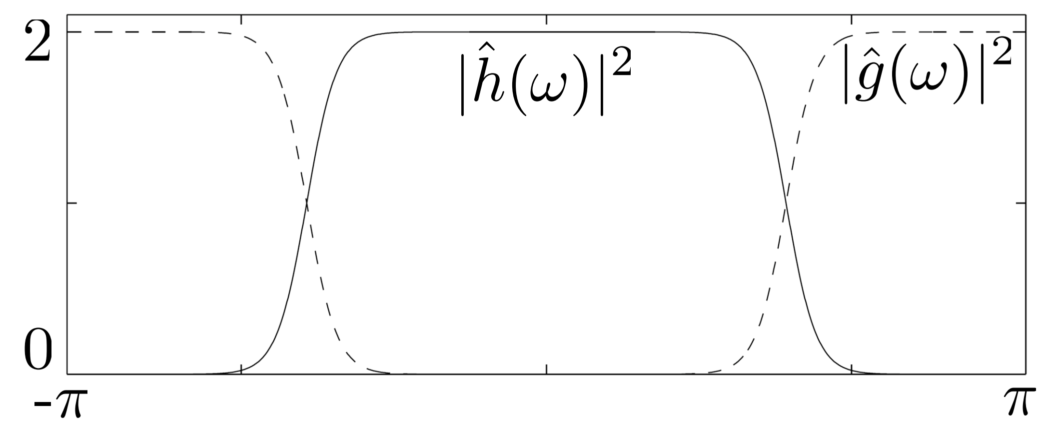


Quadrature mirror filters: $\quad g[\omega] = e^{-i\omega}\hat{h}[\omega+\pi]^* \iff g[n] = (-1)^{1-n}h[1-n]$

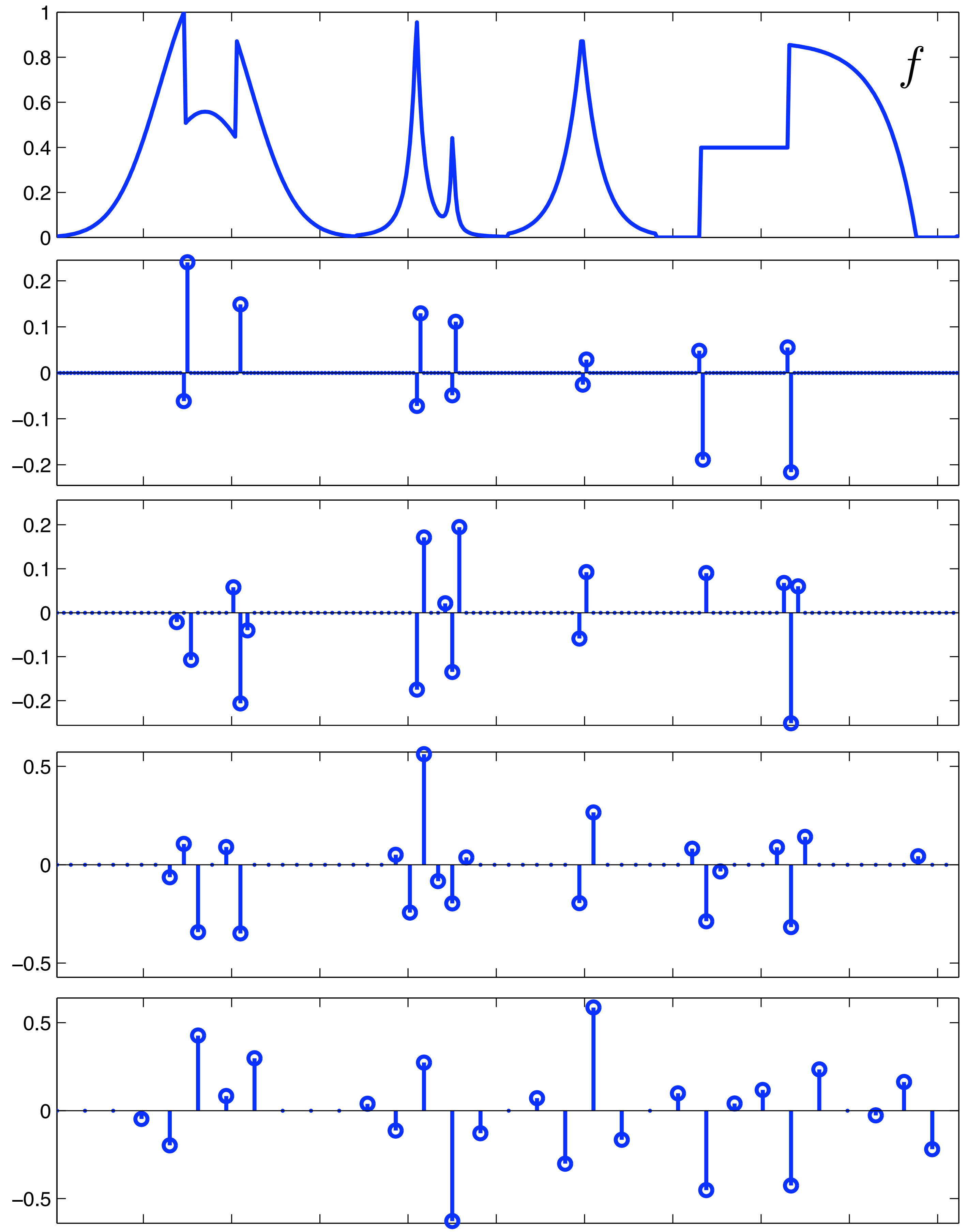
QM filter satisfy (C_4)+(C_5)

Vanishing Moment Constraint

Smooth areas: vanishing moments:

$$\forall\, k \leqslant p-1, \quad \int_0^1 \psi(x)\, x^k \mathrm{d}x = 0.$$

$$\Longrightarrow \quad \langle f,\, \psi_{j,n} \rangle \approx 0 \text{ if } f \text{ is } \mathrm{C}^\alpha,\ \alpha < p$$
$$\text{on } \mathrm{Supp}(\psi_{j,n})$$



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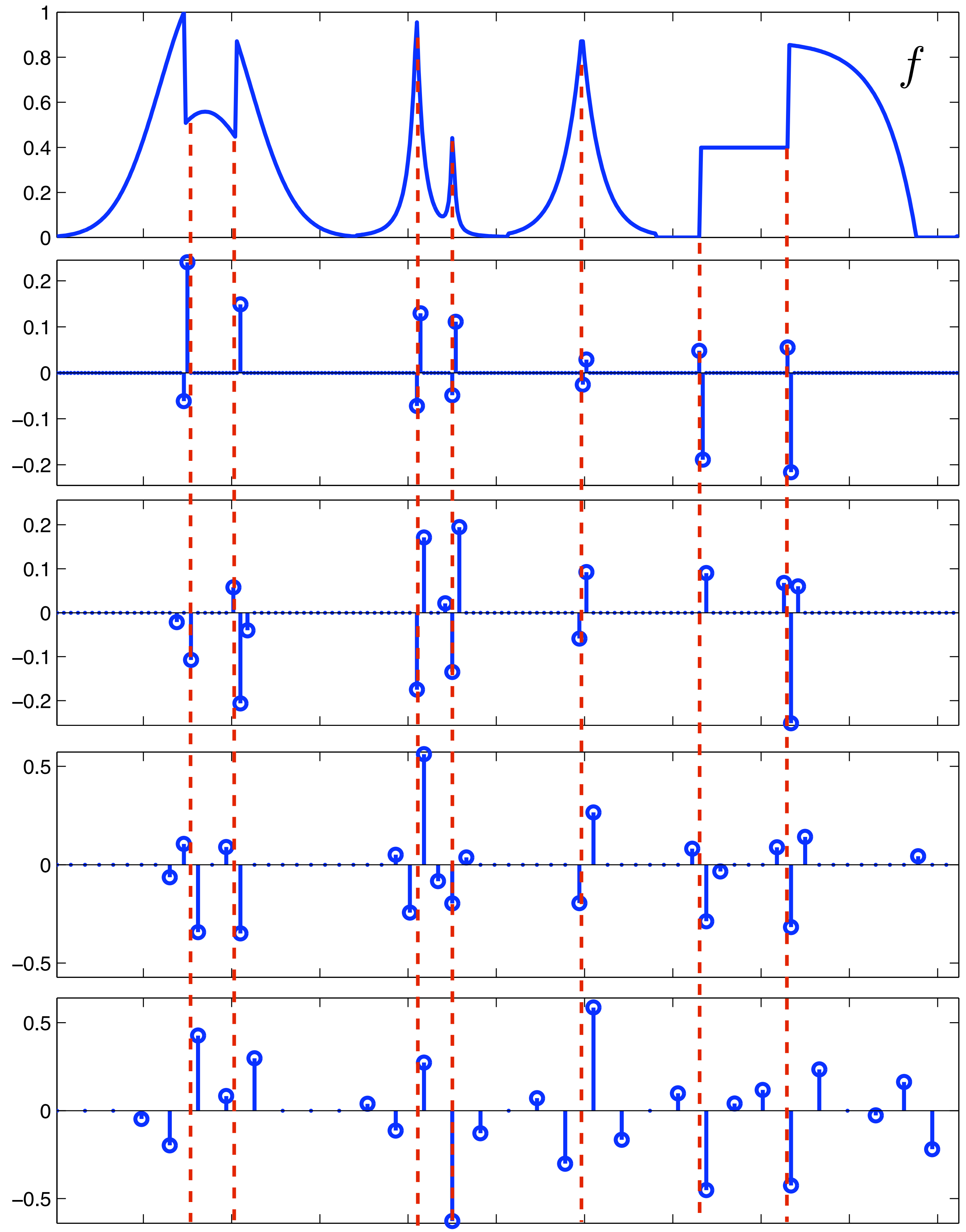
Near singularities: $\mathrm{Supp}(\psi_{j,n})$ small

$\Longrightarrow$ few large coefficients near singularities.

Theorem: $\mathrm{Supp}(\psi_{j,n})$ is larger than $2p-1$.



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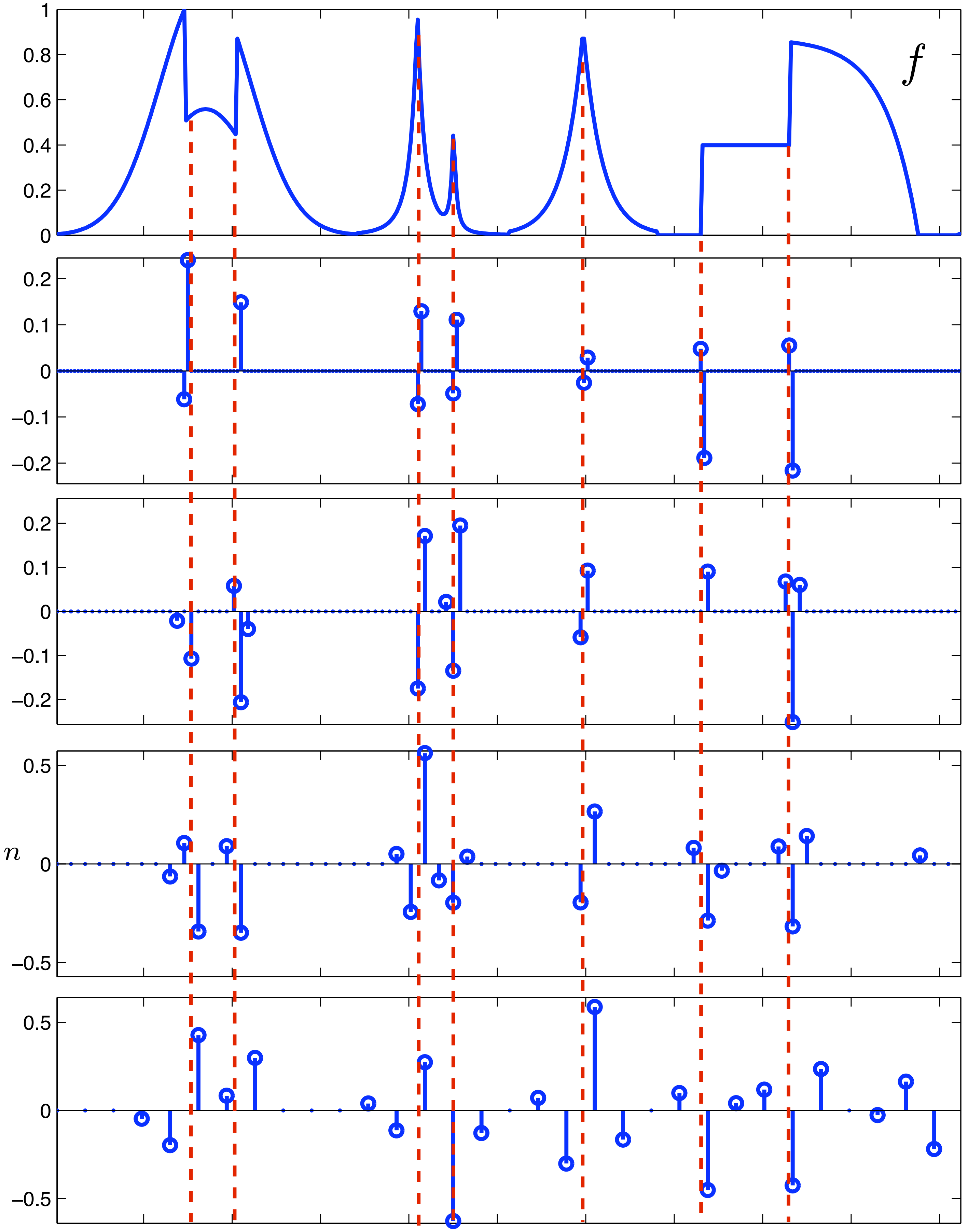
Smoothness of ψ: $\displaystyle f_M = \sum_{(j,n)\in I_M} \langle f,\, \psi_{j,n} \rangle \psi_{j,n}$

f_M same smoothness as ψ.

$\Longrightarrow$ only for cosmetic reason

Heuristic: increasing p increases the smoothness of ψ.



Daubechies Family

Compute h that satisfies:

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2$$

$$\hat{h}(0) = \sqrt{2}$$

$$p \text{ VM} \iff \forall\, k < p,\ \frac{\mathrm{d}^k \hat{h}}{\mathrm{d}\omega^k}[\pi] = 0$$

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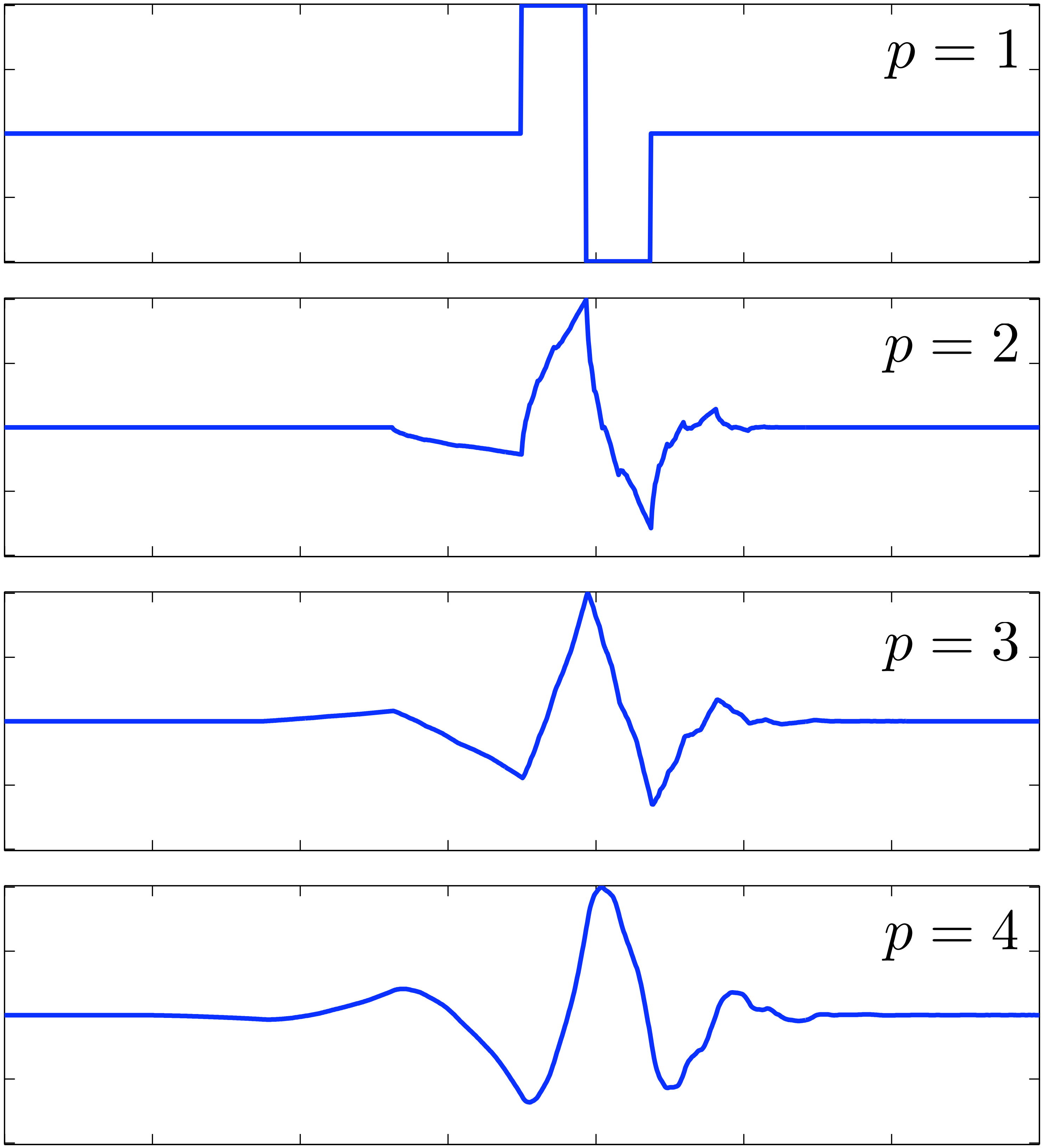
Daubechies wavelet with p VM:

Orthogonal wavelets.

Minimal support $2p-1$.



Filter $h[n]$: $n = 0$

$p = 1$ (Haar): $[0.7071; 0.7071]$

$p = 2:\ [0.4830; 0.8365; 0.2241; -0.1294]$

$n = 0$ $n = 0$

$p = 3:\ [0; 0.3327; 0.8069; 0.4599;$
$-0.1350; -0.0854; 0.0352]$

Overview

- Multiresolutions
- Fast Wavelet Transform
- Wavelet Design
- 2D Wavelet Processing

Anisotropic Wavelet Transform

Separable basis: $\psi_{j_1,j_2,n_1,n_2}(x) = \psi_{j_1,n_1}(x_1)\psi_{j_2,n_2}(x_2)$

Orthogonal basis of $\mathrm{L}^2(\mathbb{R}^2)$.

Discretization: orthogonal basis of $\mathbb{C}^{N\times N}$.

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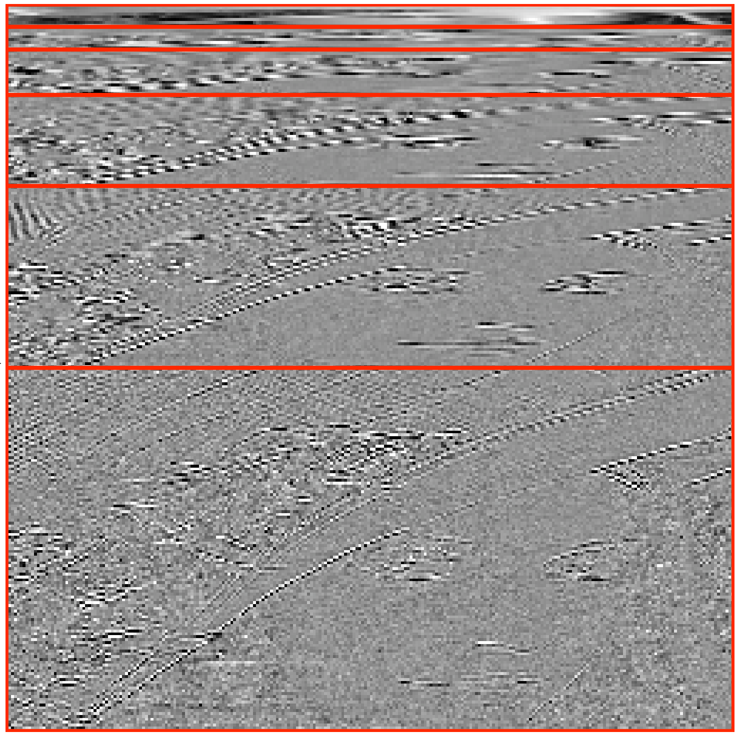
Algorithm:

- Apply 1D wavelet transform to each column.



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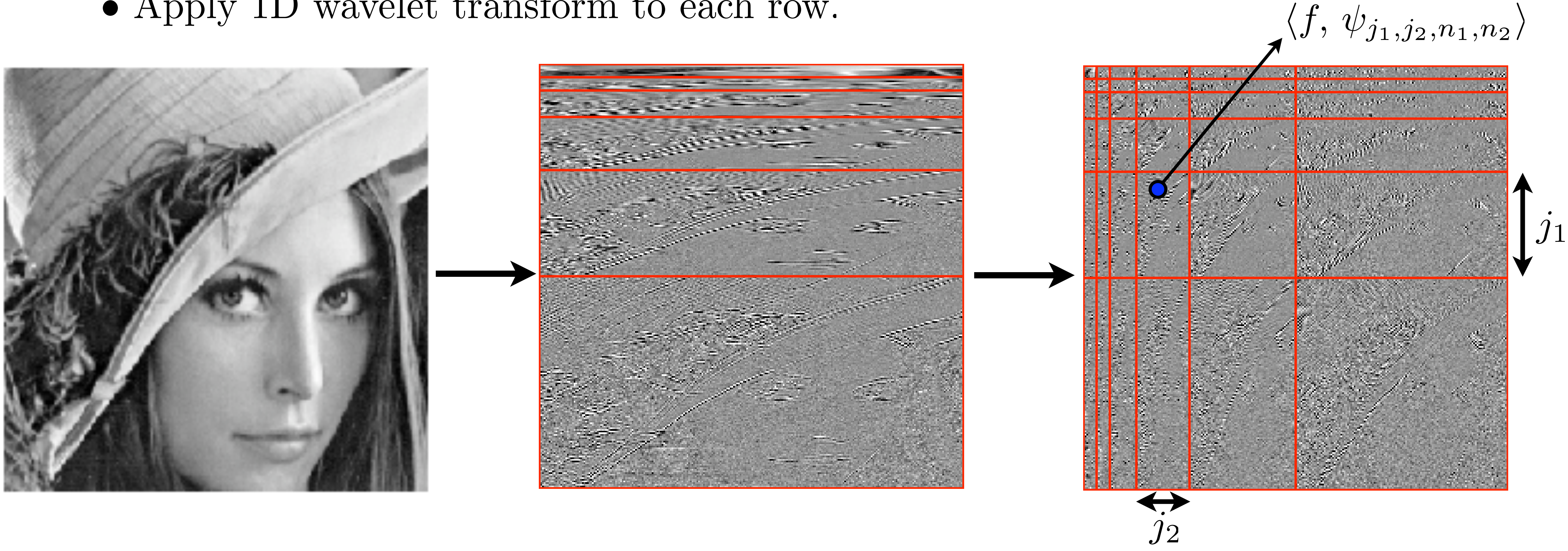
- Apply 1D wavelet transform to each row.



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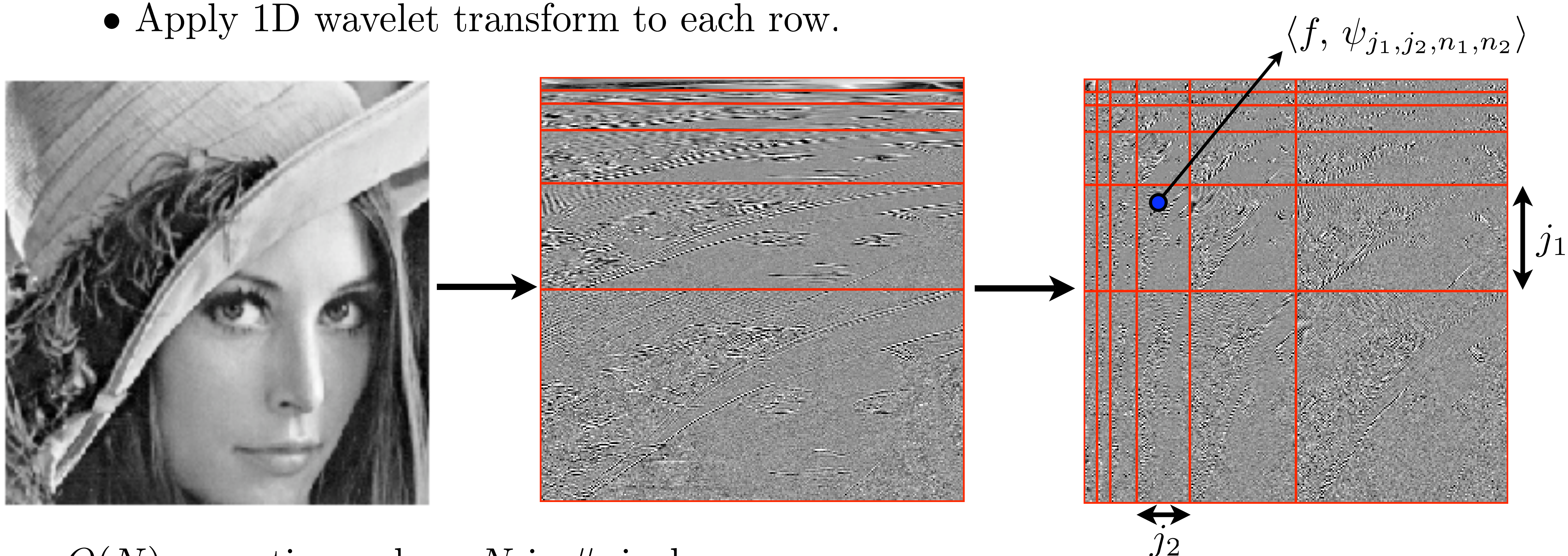

$O(N)$ operations where N is #pixels.

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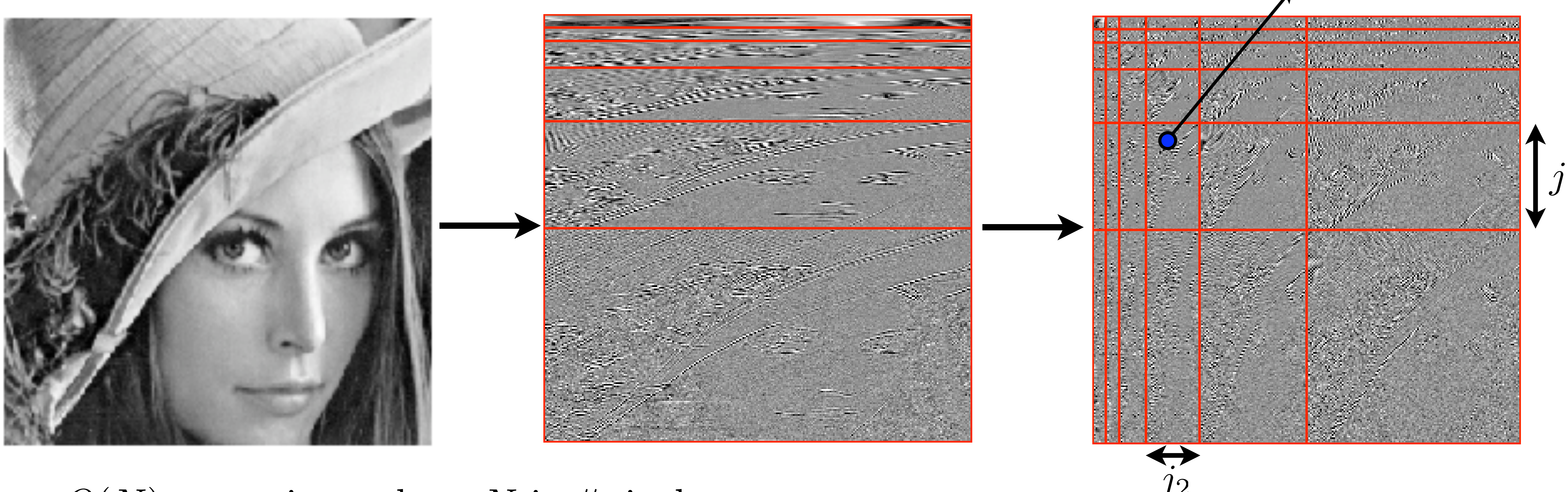

$O(N)$ operations where N is #pixels.

Favors horizontal/vertical axes.
$\rightarrow$ axis-aligned approximation artifacts.
$\rightarrow$ not suitable for natural images.

2D Multi-resolutions

Separable approximation spaces: $V_j \otimes V_j = \mathrm{Span}\{\varphi_{j,n_1}(x_1)\varphi_{j,n_2}(x_2)\}_{n_1,n_2}$

$$\mathrm{L}^2(\mathbb{R}^2) \longrightarrow \cdots \longrightarrow V_{j-1} \otimes V_{j-1} \longrightarrow V_j \otimes V_j \longrightarrow V_{j+1} \otimes V_{j+1} \longrightarrow \cdots \longrightarrow \{0\}$$

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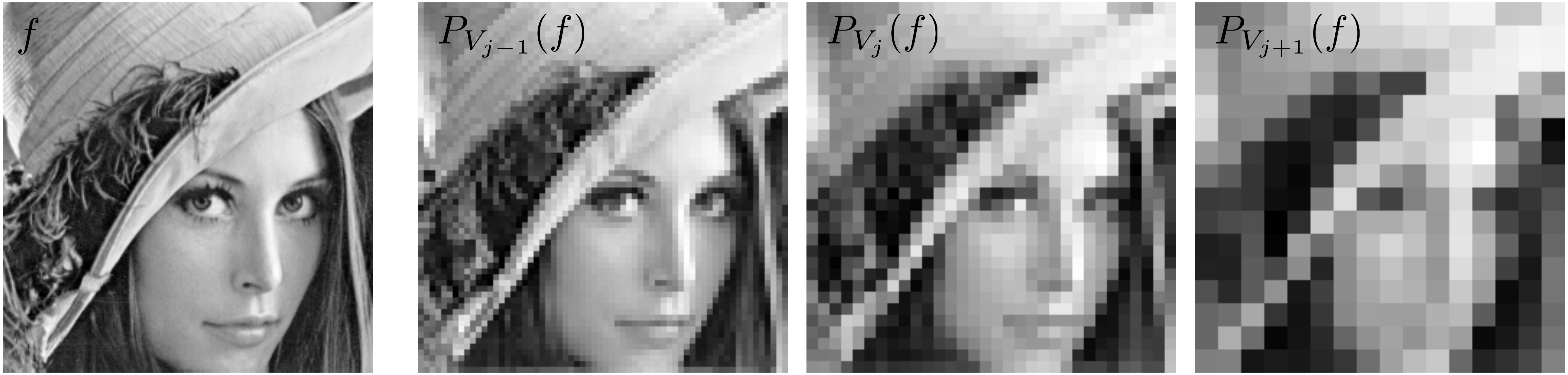
2D Piecewise-constant Haar approximation:



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Separable approximation spaces: $V_j \otimes V_j = \mathrm{Span}\{\varphi_{j,n_1}(x_1)\varphi_{j,n_2}(x_2)\}_{n_1,n_2}$

$$\begin{tikzcd}
\mathrm{L}^2(\mathbb{R}^2) \arrow[r] & \cdots \arrow[r] \arrow[dr] & V_{j-1}\otimes V_{j-1} \arrow[r] \arrow[dr] & V_j \otimes V_j \arrow[r] \arrow[dr] & V_{j+1}\otimes V_{j+1} \arrow[r] \arrow[dr] & \cdots \arrow[r] & \{0\} \\
 & & W_{j-1}^{(2)} & W_j^{(2)} & W_{j+1}^{(2)} & &
\end{tikzcd}$$

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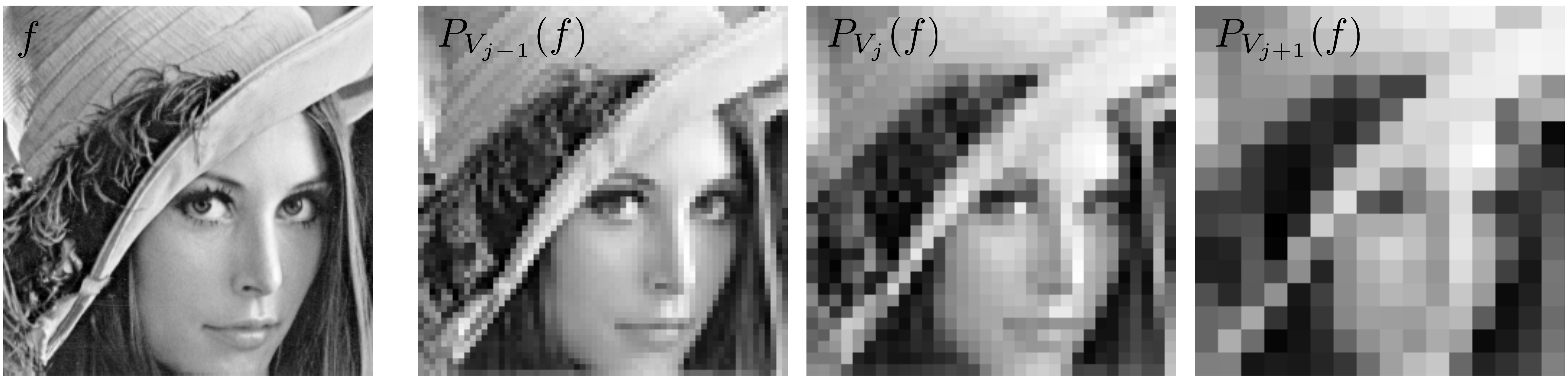


$$V_{j-1} \otimes V_{j-1} = (V_j \oplus^\perp W_j) \otimes (V_j \oplus^\perp W_j)$$

$$= \underbrace{(V_j \otimes V_j)}_{\text{Coarse approximation}} \oplus \underbrace{\underbrace{(V_j \otimes W_j)}_{\text{Vertical details}} \oplus \underbrace{(W_j \otimes V_j)}_{\text{Horizontal details}} \oplus \underbrace{(W_j \otimes W_j)}_{\text{Diagonal details}}}_{W_j^{(2)}}$$

2D Wavelet Basis

$$V_{j-1} \otimes V_{j-1} = (V_j \otimes V_j) \oplus W_j^{(2)} \qquad W_j^{(2)} = (V_j \otimes W_j) \oplus (W_j \otimes V_j) \oplus (W_j \otimes W_j)$$

$$\mathrm{V}_j \oplus W_j = \mathrm{Span}\{\varphi_{j,n_1}(x_1)\psi_{j,n_2}(x_2)\}_{n_1,n_2} = \mathrm{Span}\{\psi^V_{j,n_1,n_2}\}_{n_1,n_2}$$

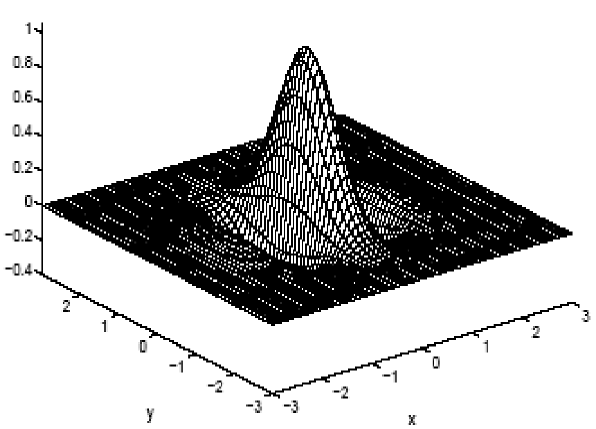
$$\mathrm{W}_j \oplus V_j = \mathrm{Span}\{\psi_{j,n_1}(x_1)\varphi_{j,n_2}(x_2)\}_{n_1,n_2} = \mathrm{Span}\{\psi^H_{j,n_1,n_2}\}_{n_1,n_2}$$

$$\mathrm{W}_j \oplus W_j = \mathrm{Span}\{\psi_{j,n_1}(x_1)\psi_{j,n_2}(x_2)\}_{n_1,n_2} = \mathrm{Span}\{\psi^D_{j,n_1,n_2}\}_{n_1,n_2}$$

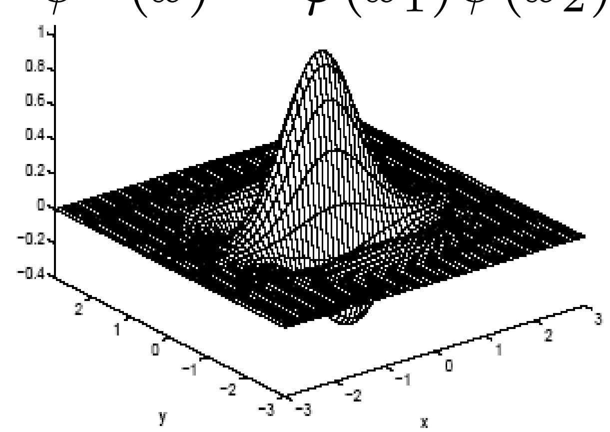
Orthogonal basis: $\left\{\psi^{\omega}_{j,n_1,n_2}\right\}^{\omega=H,V,D}_{j,n_1,n_2\in\mathbb{Z}}$

$$\psi^{\omega}_{j,n_1,n_2}(x) = \frac{1}{2^j}\psi^{\omega}\left(\frac{x_1 - 2^j n_1}{2^j}, \frac{x_2 - 2^j n_2}{2^j}\right) \qquad \omega \in \{H, V, D\}$$

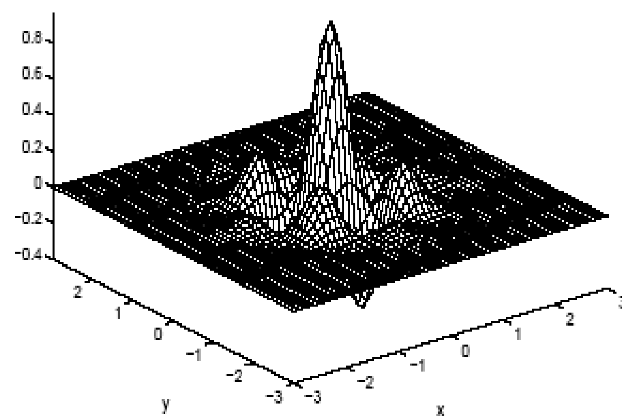
$\psi^H(x) = \psi(x_1)\varphi(x_2)$

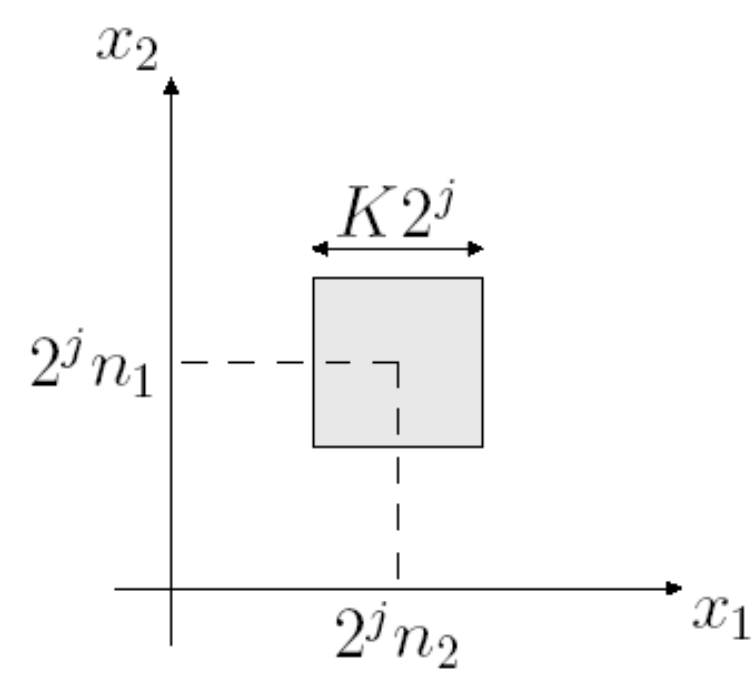


$\psi^V(x) = \varphi(x_1)\psi(x_2)$



$\psi^D(x) = \psi(x_1)\psi(x_2)$





Discrete 2D Wavelets Coefficients

Analog image $f_0 \in \mathrm{L}^2([0,1]^2)$. Discretization points $\{(n_1, n_2)2^J\}_n$ where $2^{-J} = \sqrt{N}$.

Discretized image of N pixels: for $n = (n_1, n_2)$ $f[n] = a_J[n] = \langle f_0, \varphi_{J,n} \rangle$

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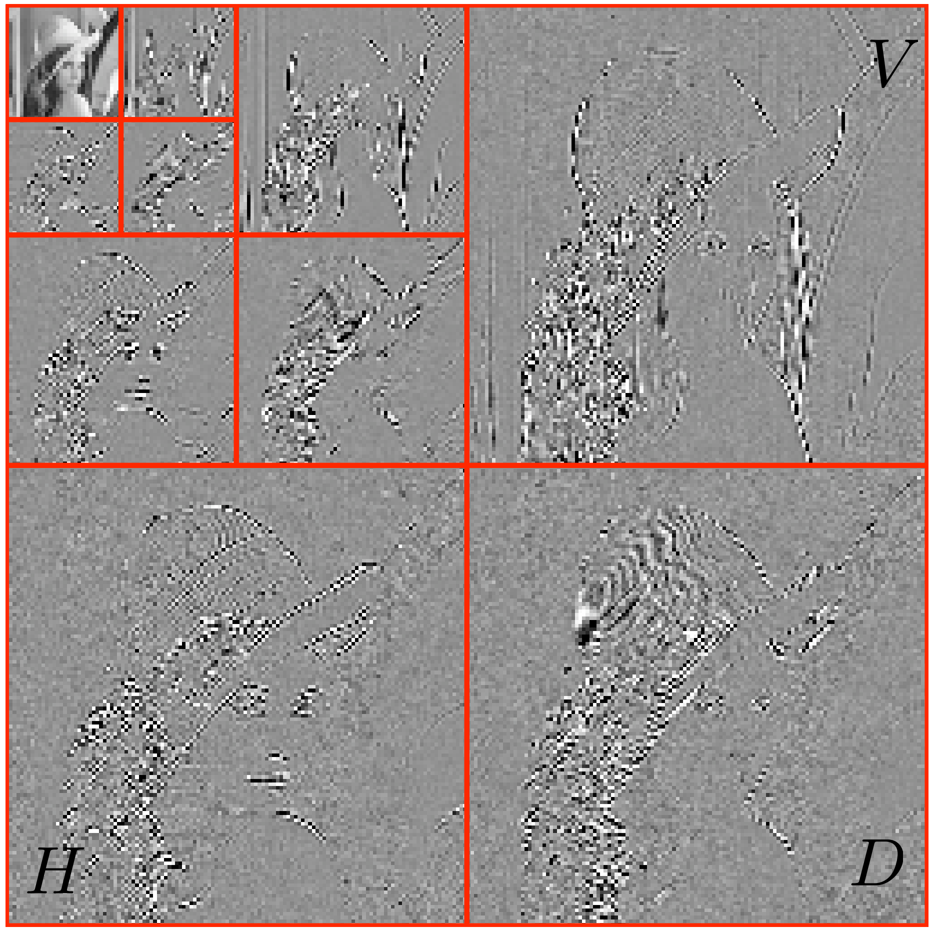
Discrete wavelet basis $\{\bar\psi_{j,n}^\omega\}_{j,n}^\omega$ of $\mathbb{C}^N$, for $\begin{cases} J < j \leqslant 0 \\ 0 \leqslant n_1, n_2 < 2^{-j} \\ \omega \in \{H, V, D\} \end{cases}$



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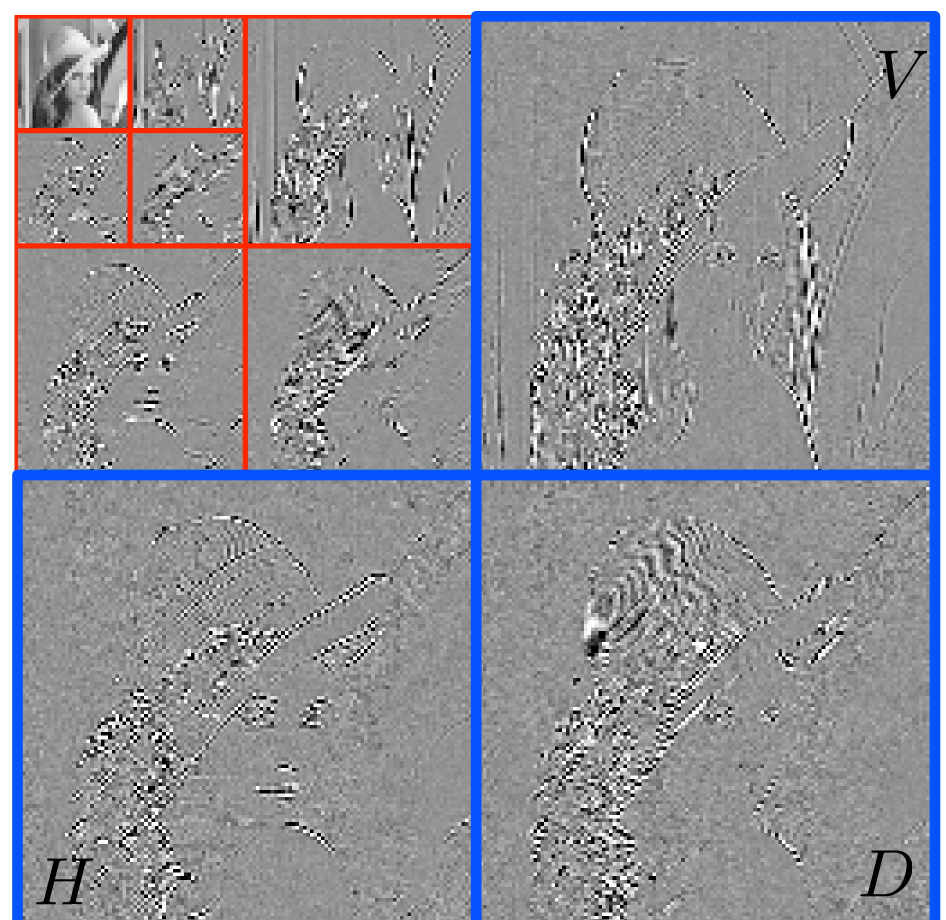

$j = -8$

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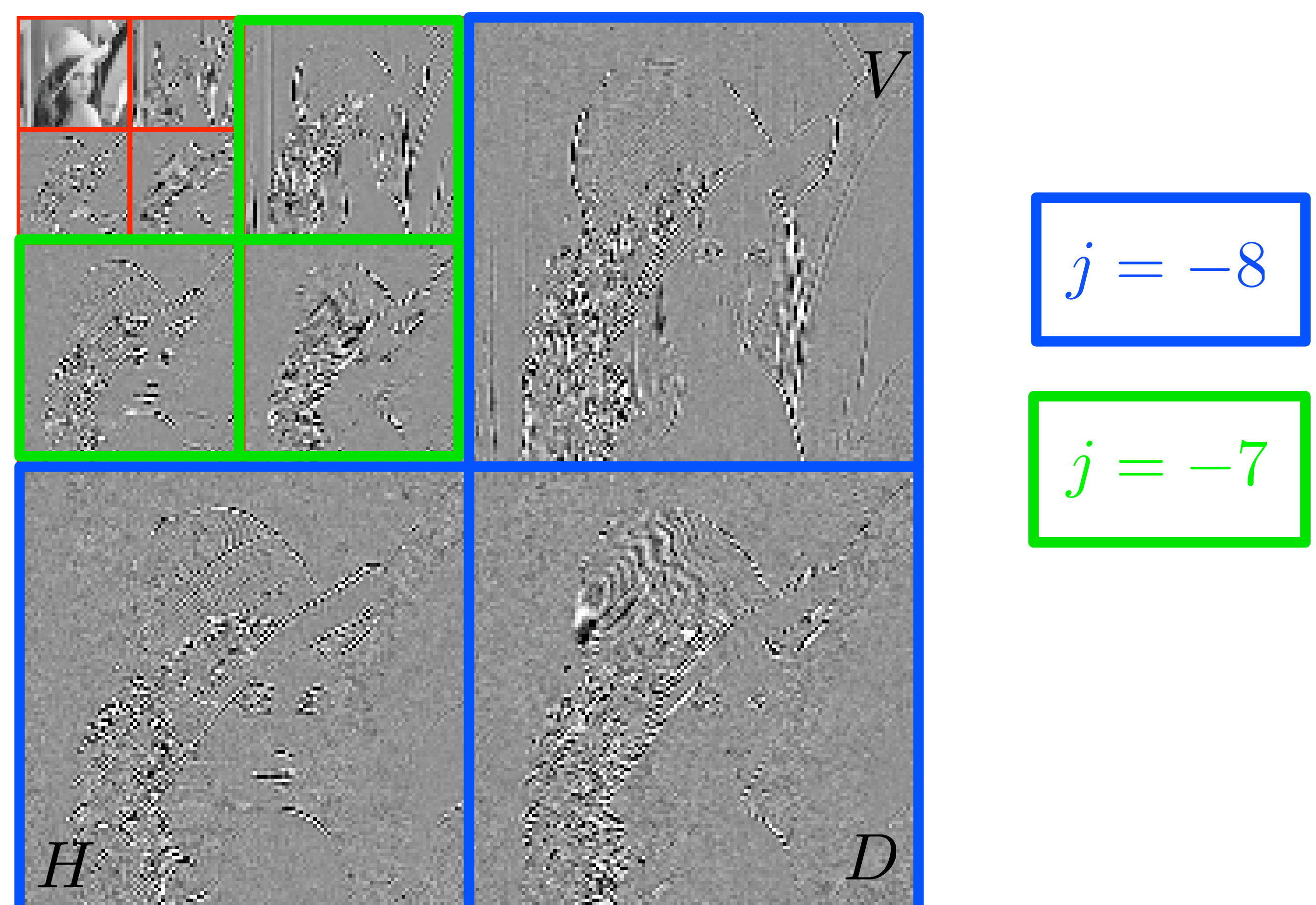

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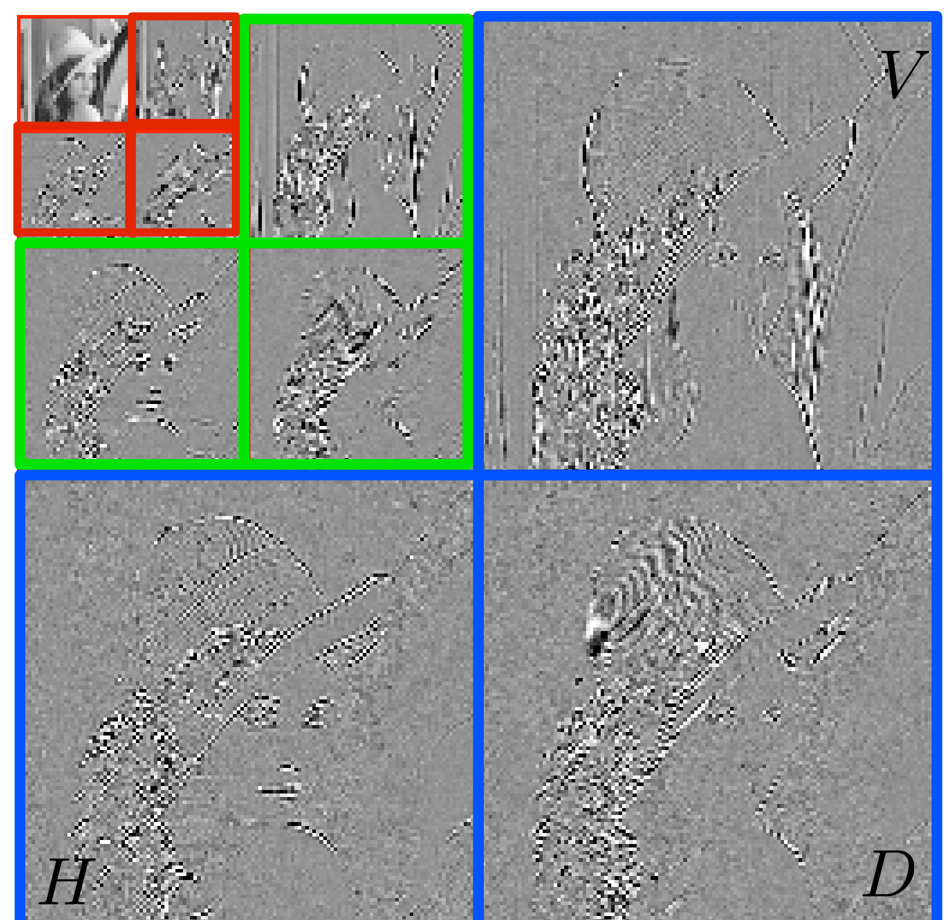
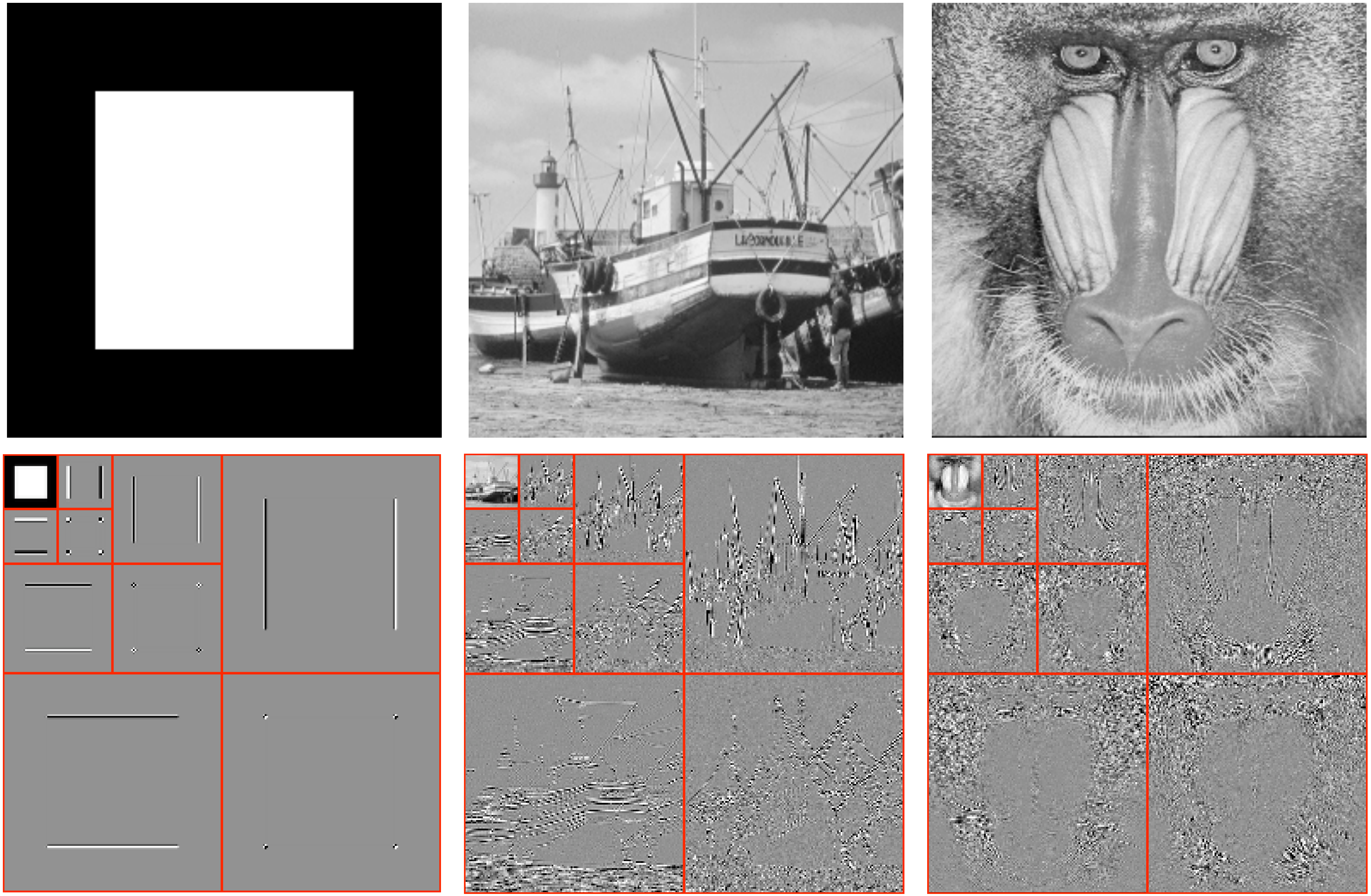


$j = -8$

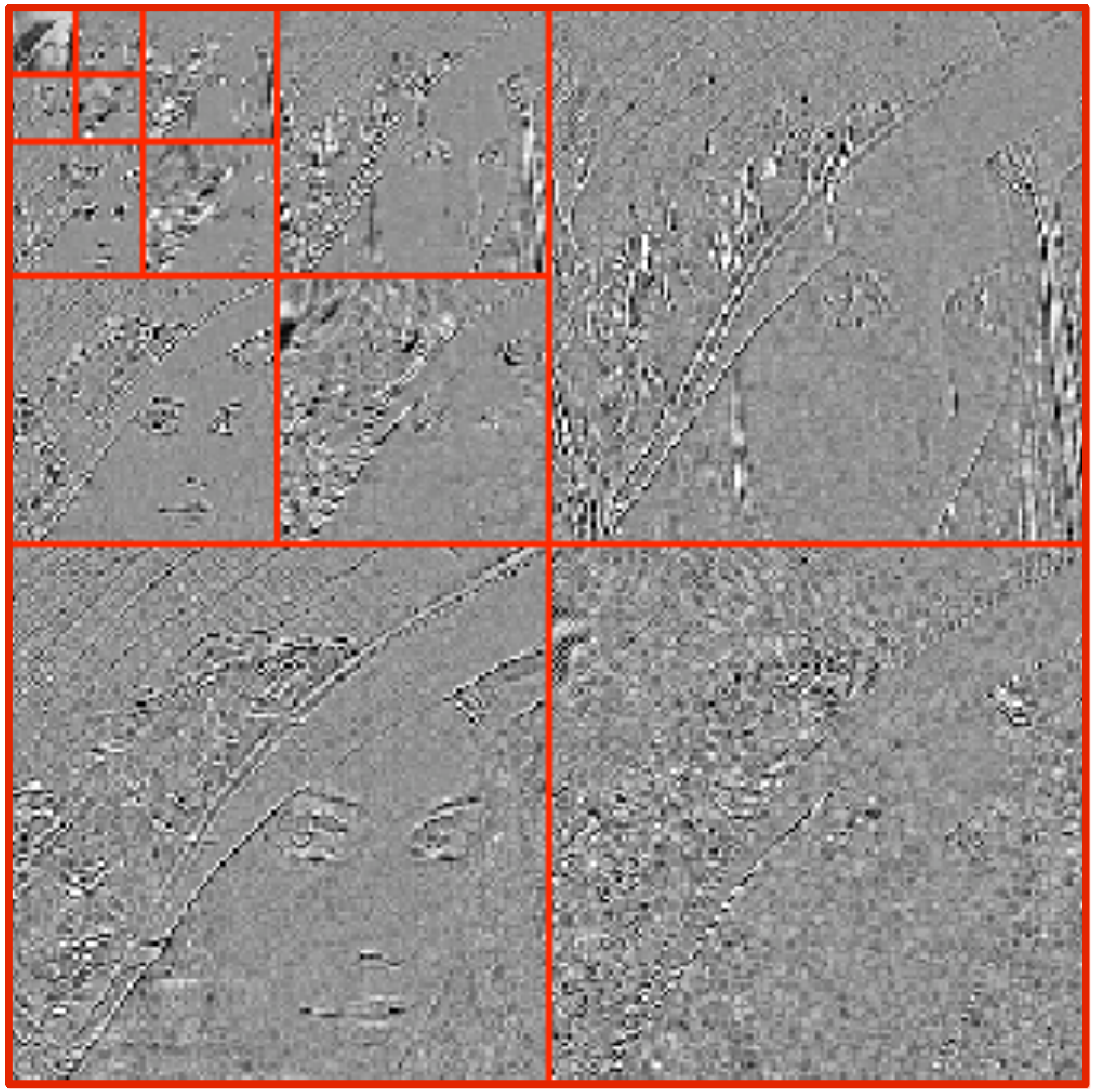
$j = -7$

$j = -6$

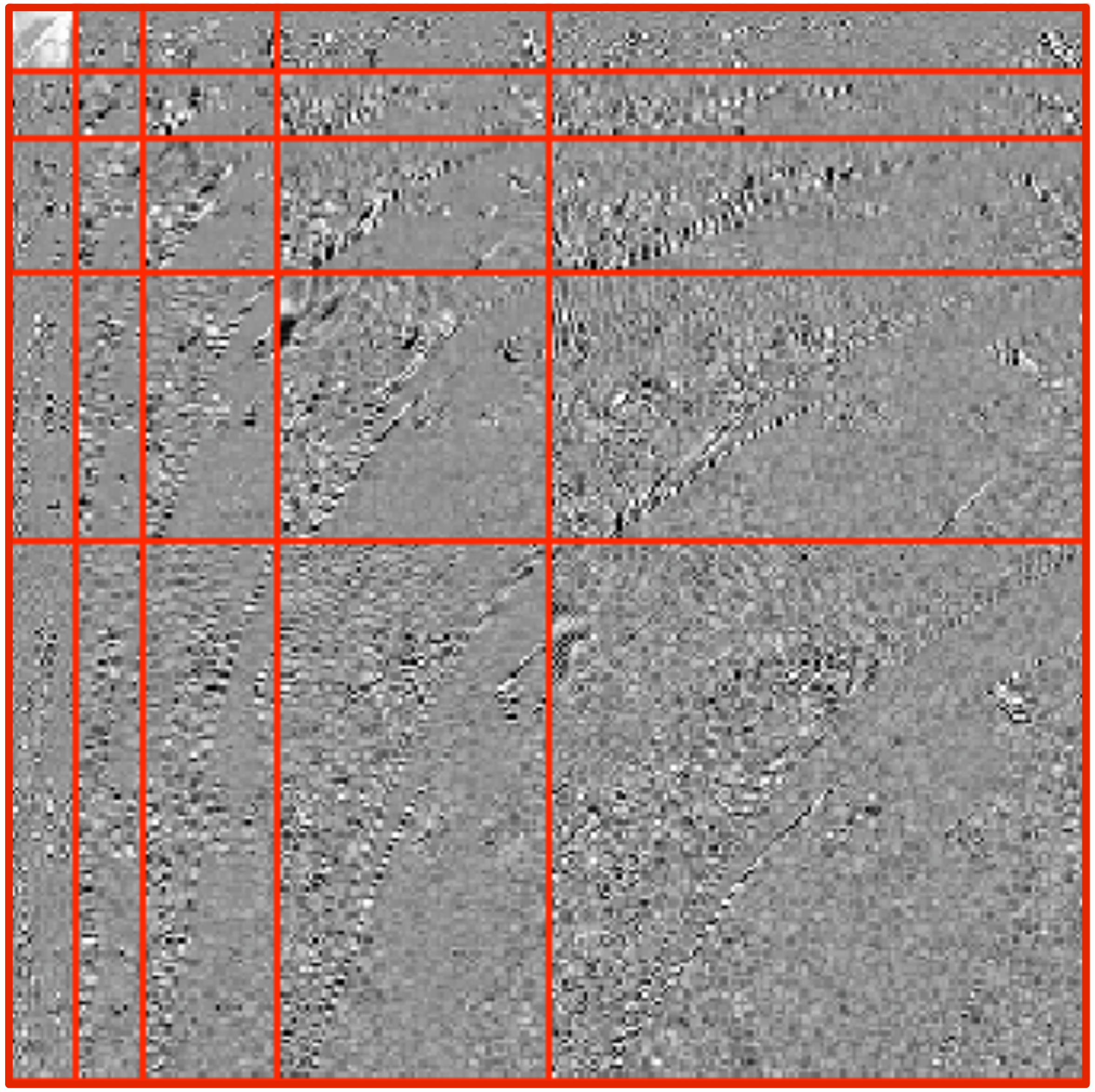
Examples of Decompositions



Separable vs. Isotropic

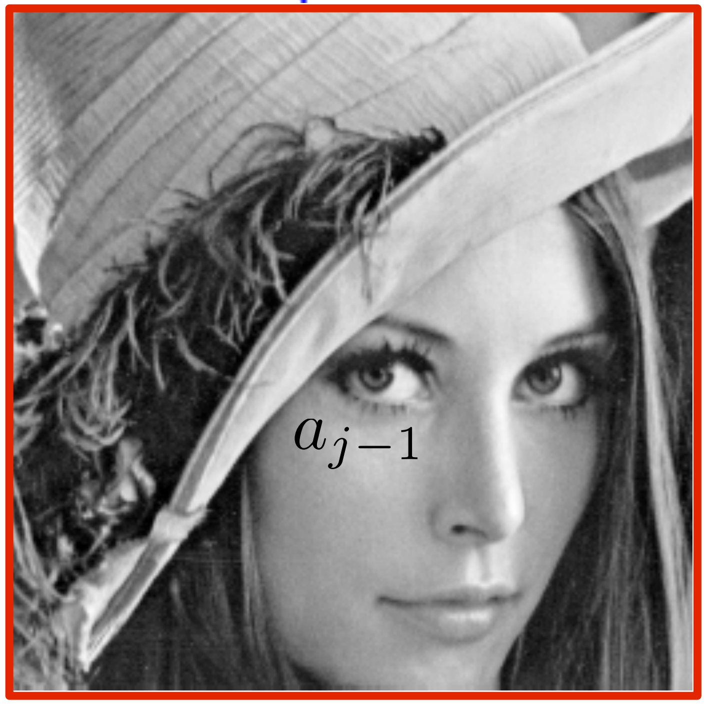


Isotropic, $\langle f,\, \psi^{k}_{j,n_1,n_2} \rangle$

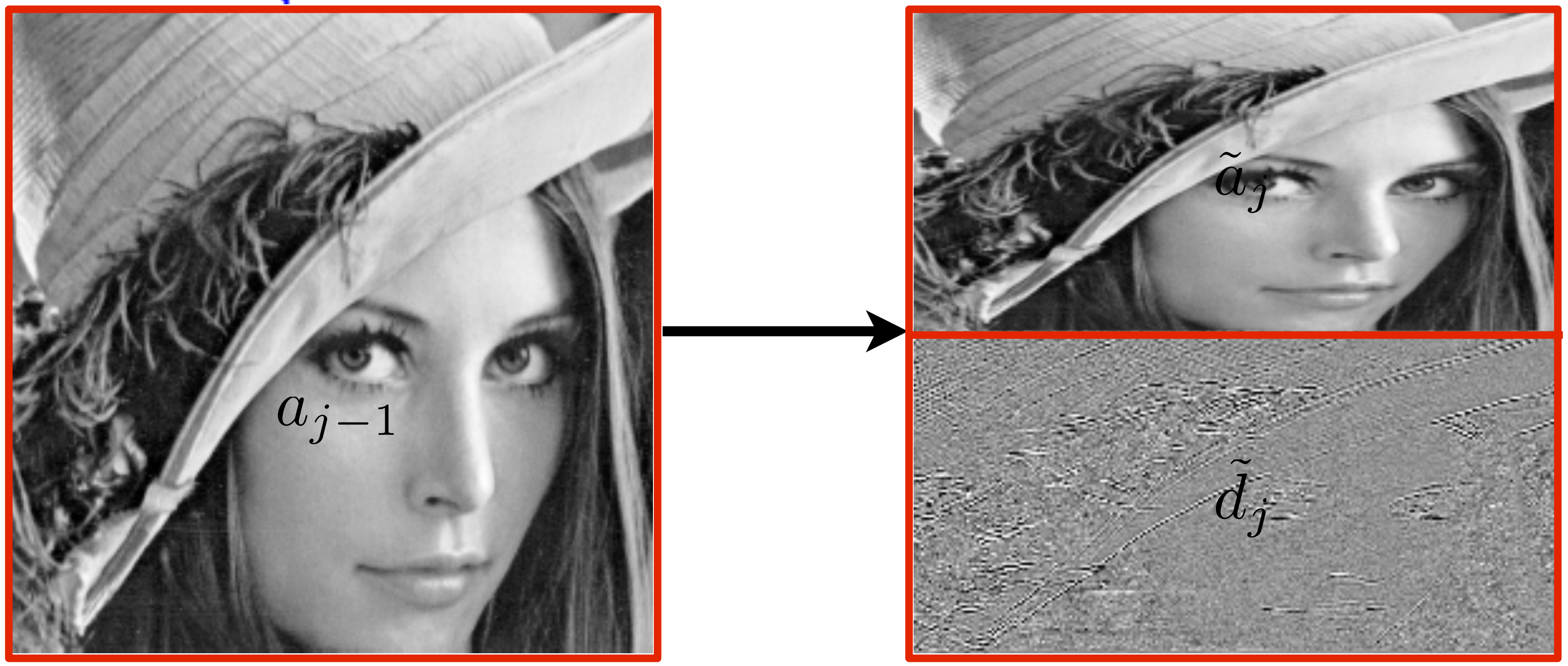


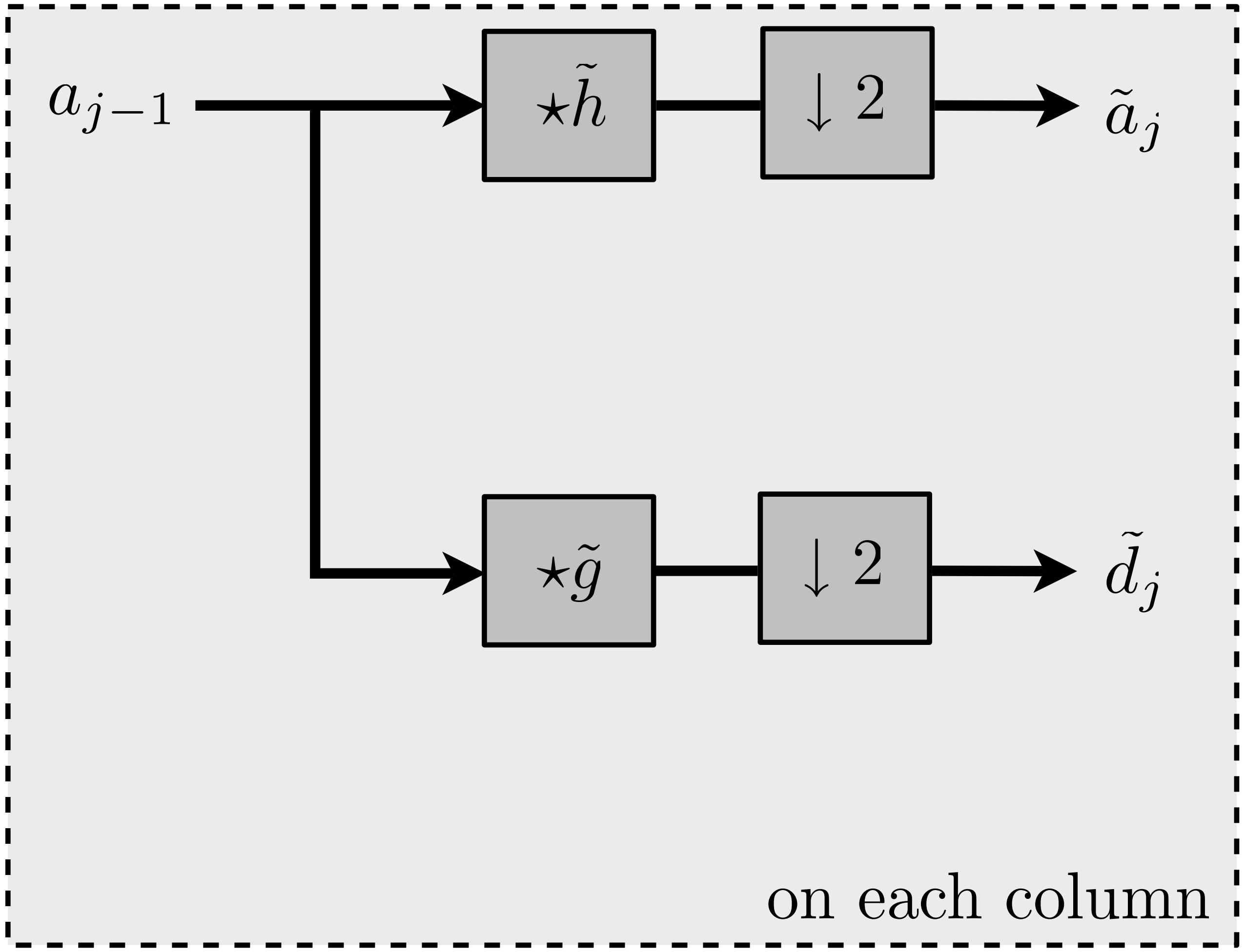
Separable, $\langle f,\, \psi_{j_1,j_2,n_1,n_2} \rangle$

Fast 2D Wavelet Transform



Fast 2D Wavelet Transform





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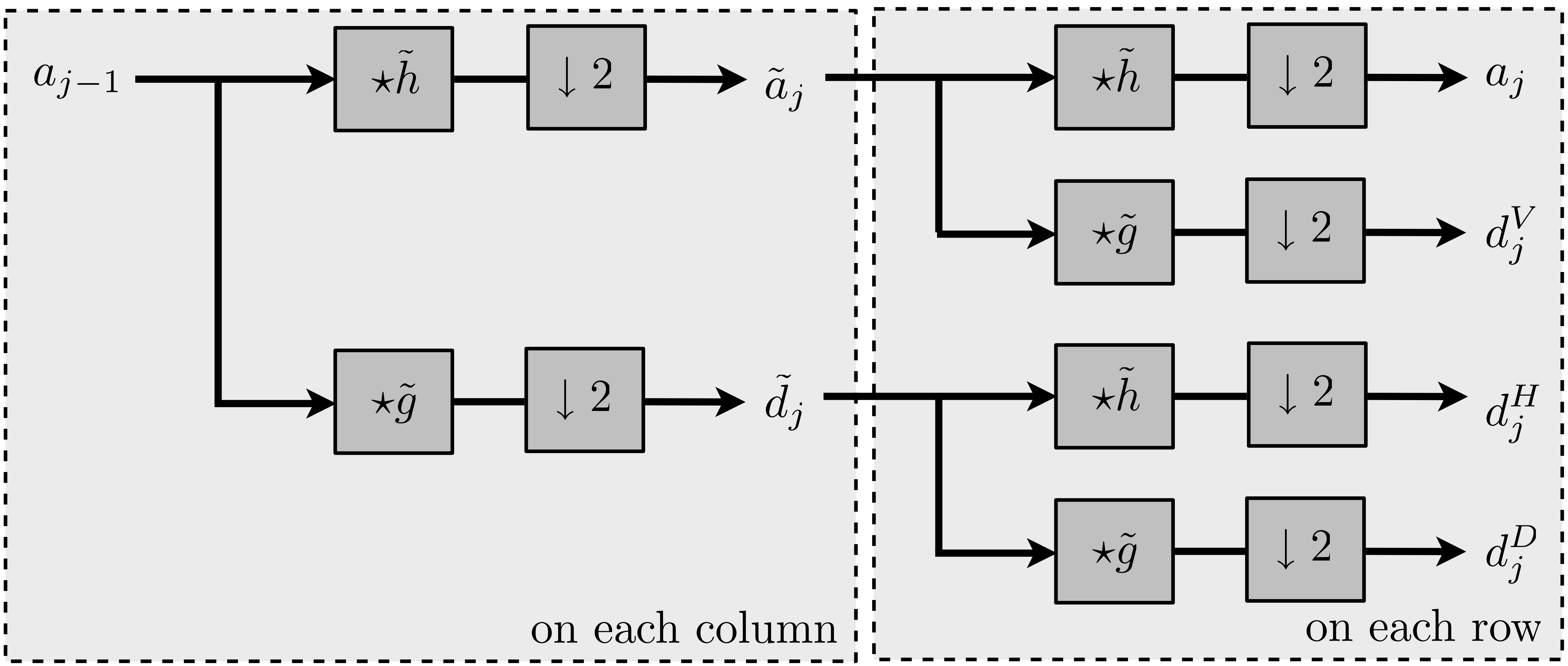
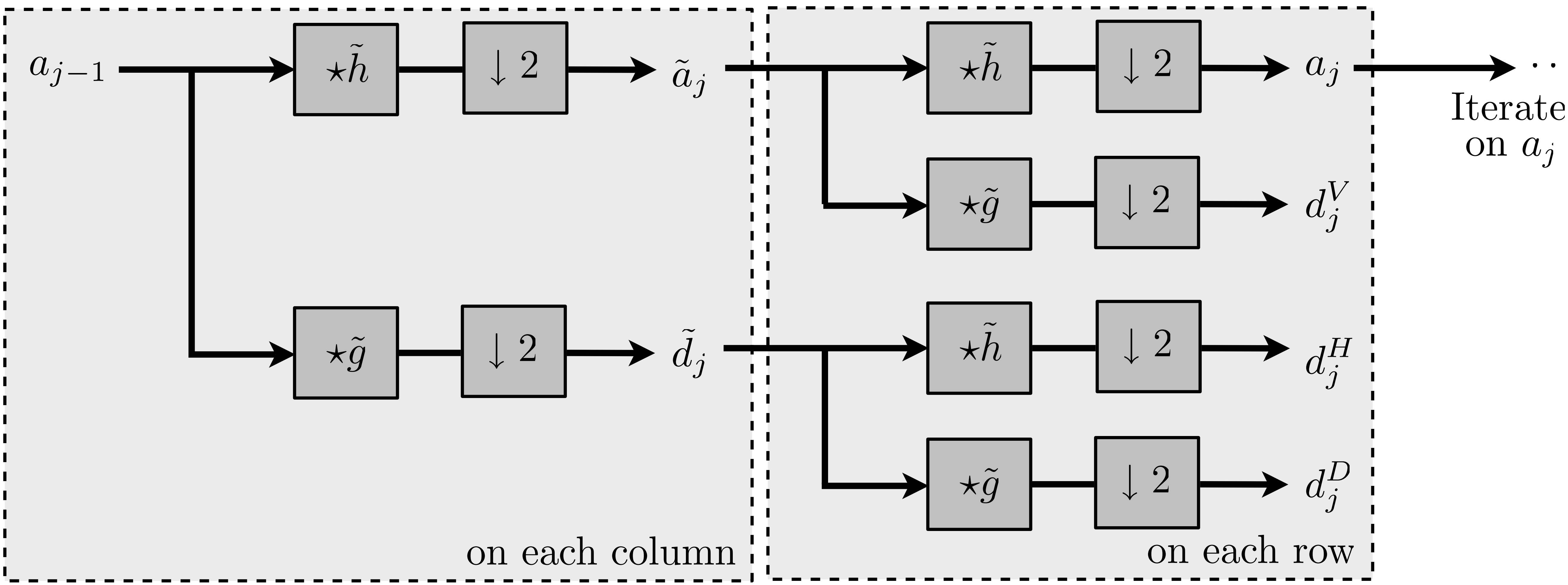
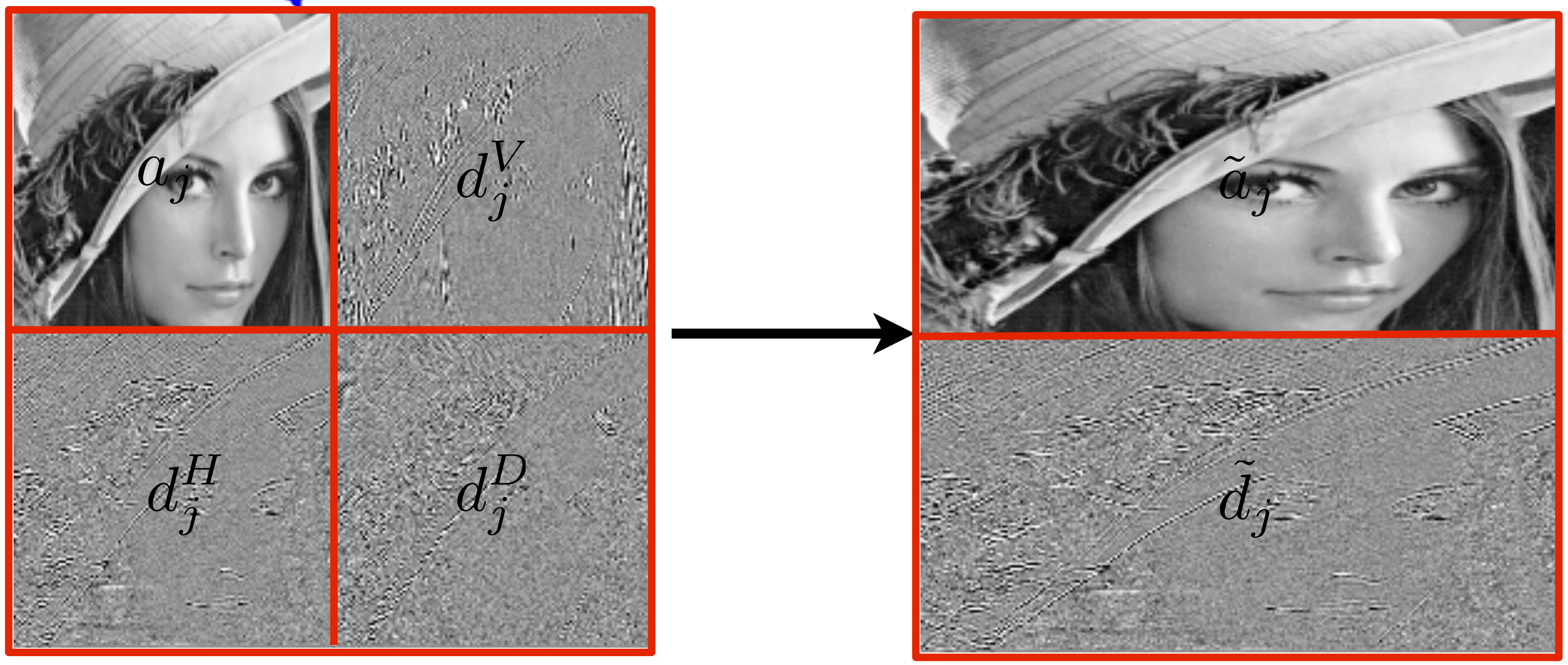
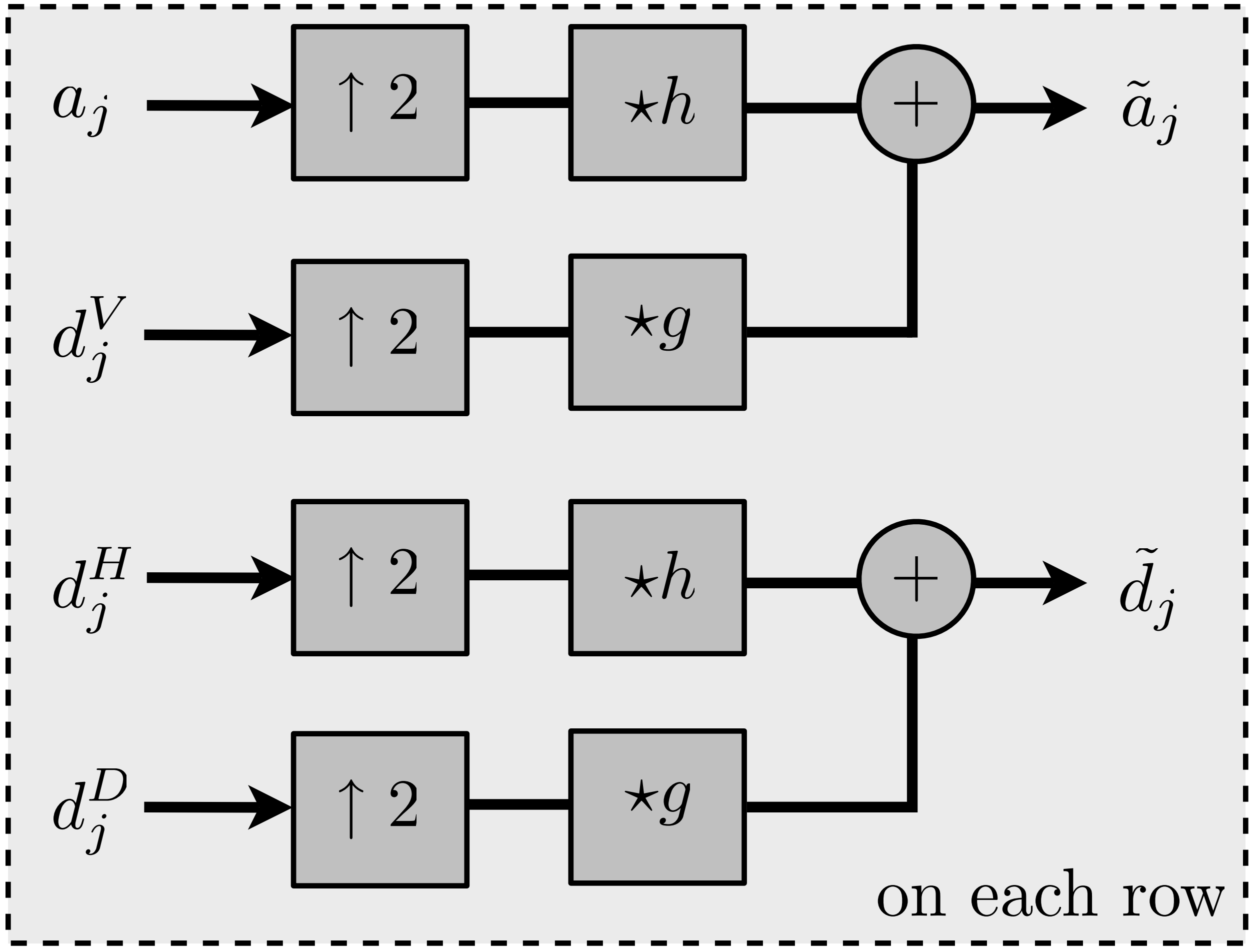


Fast 2D Wavelet Transform



Inverse 2D Wavelet Transform





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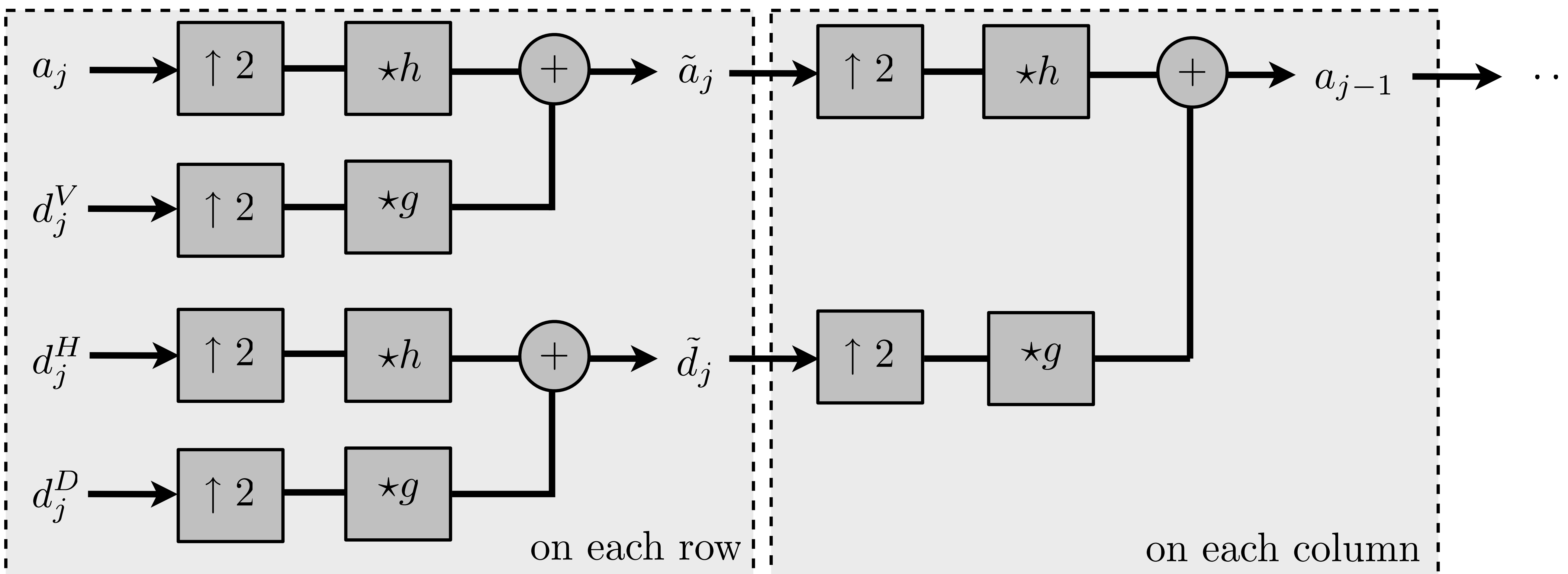
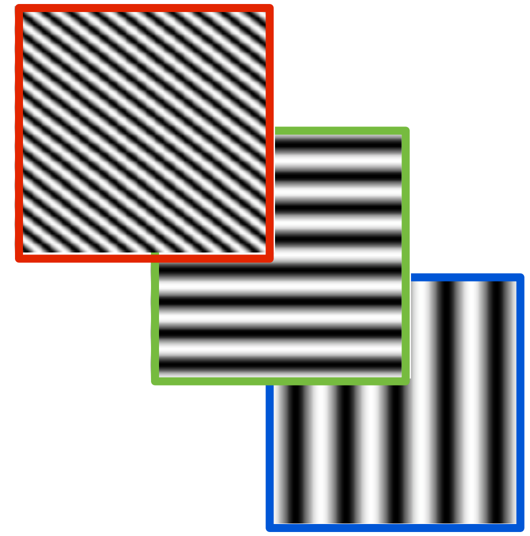
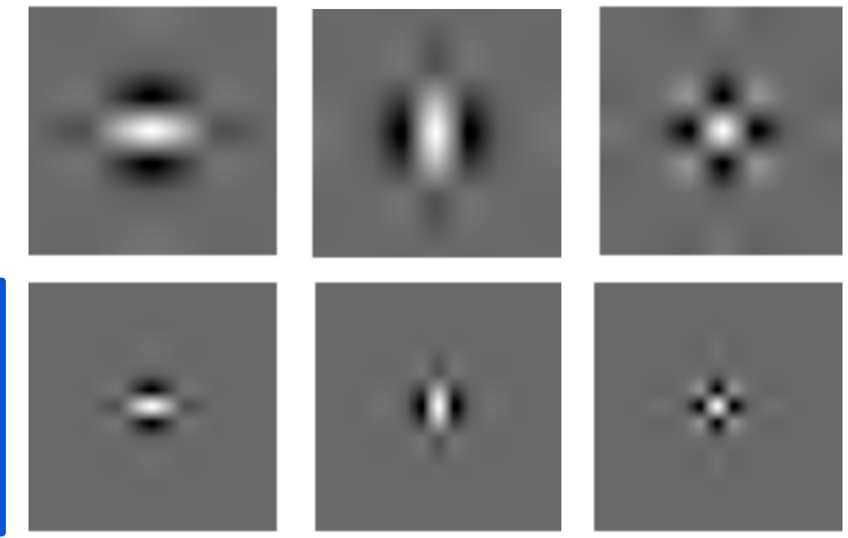


Conclusion

Orthogonal decompositions:

Fourier, wavelets, etc.





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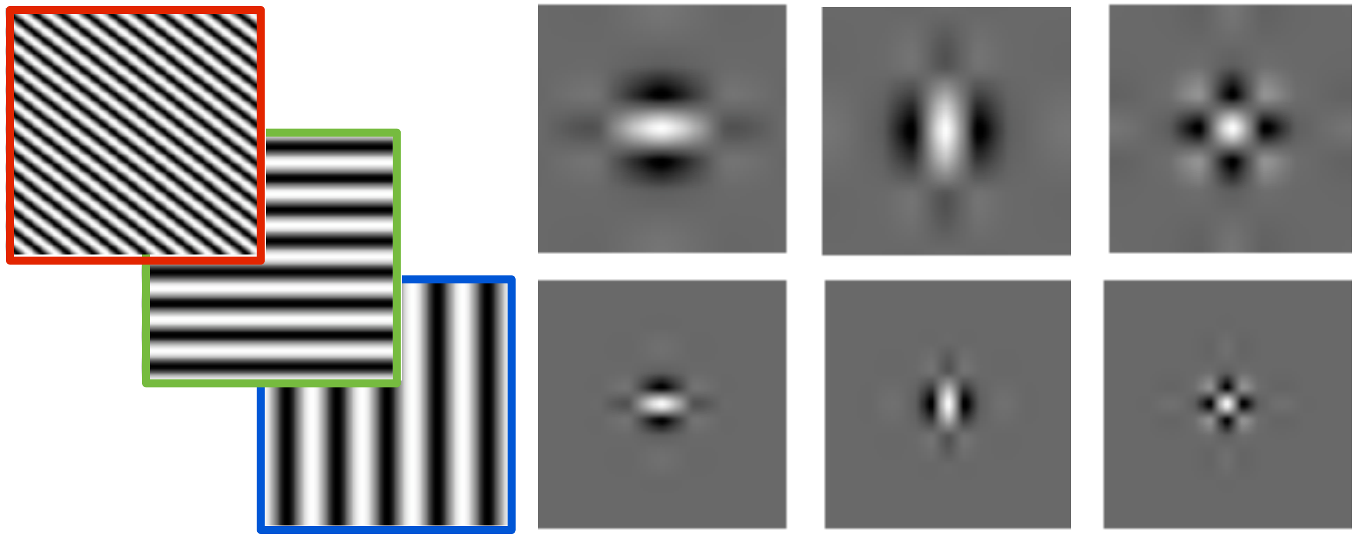
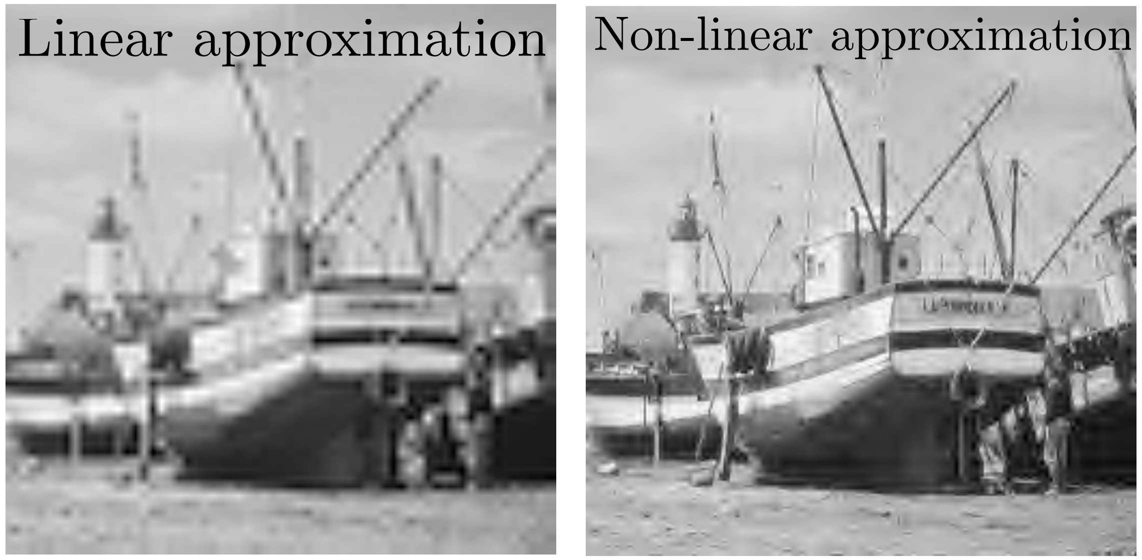
Linear vs. non-linear approximation:

Linear = filtering.

Non-linear = thresholding.





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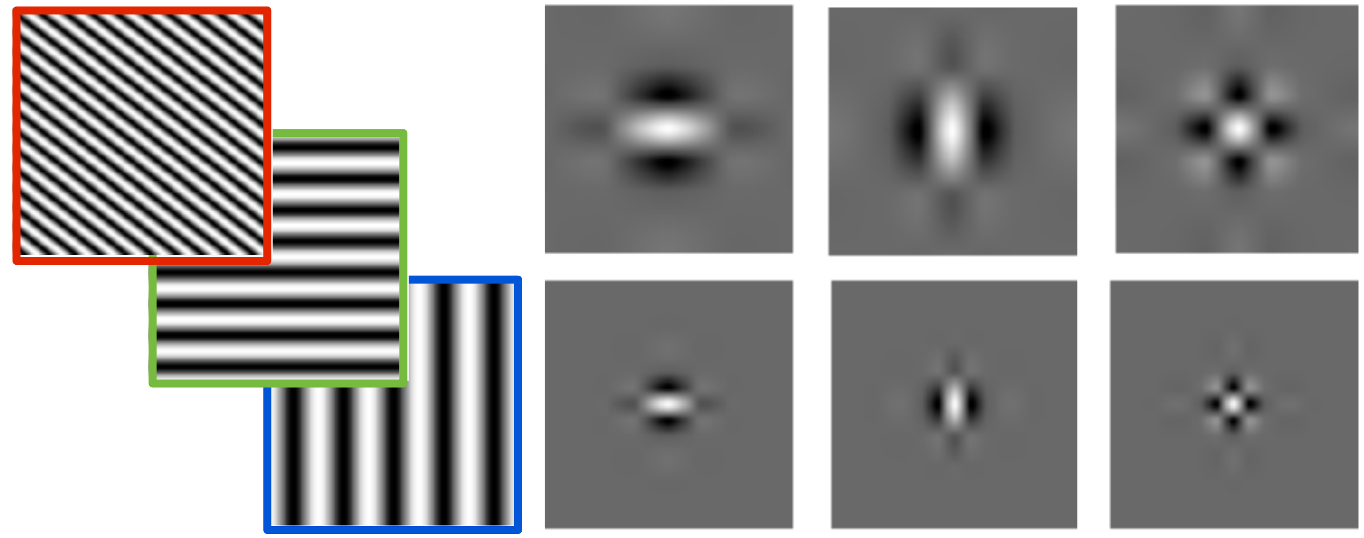
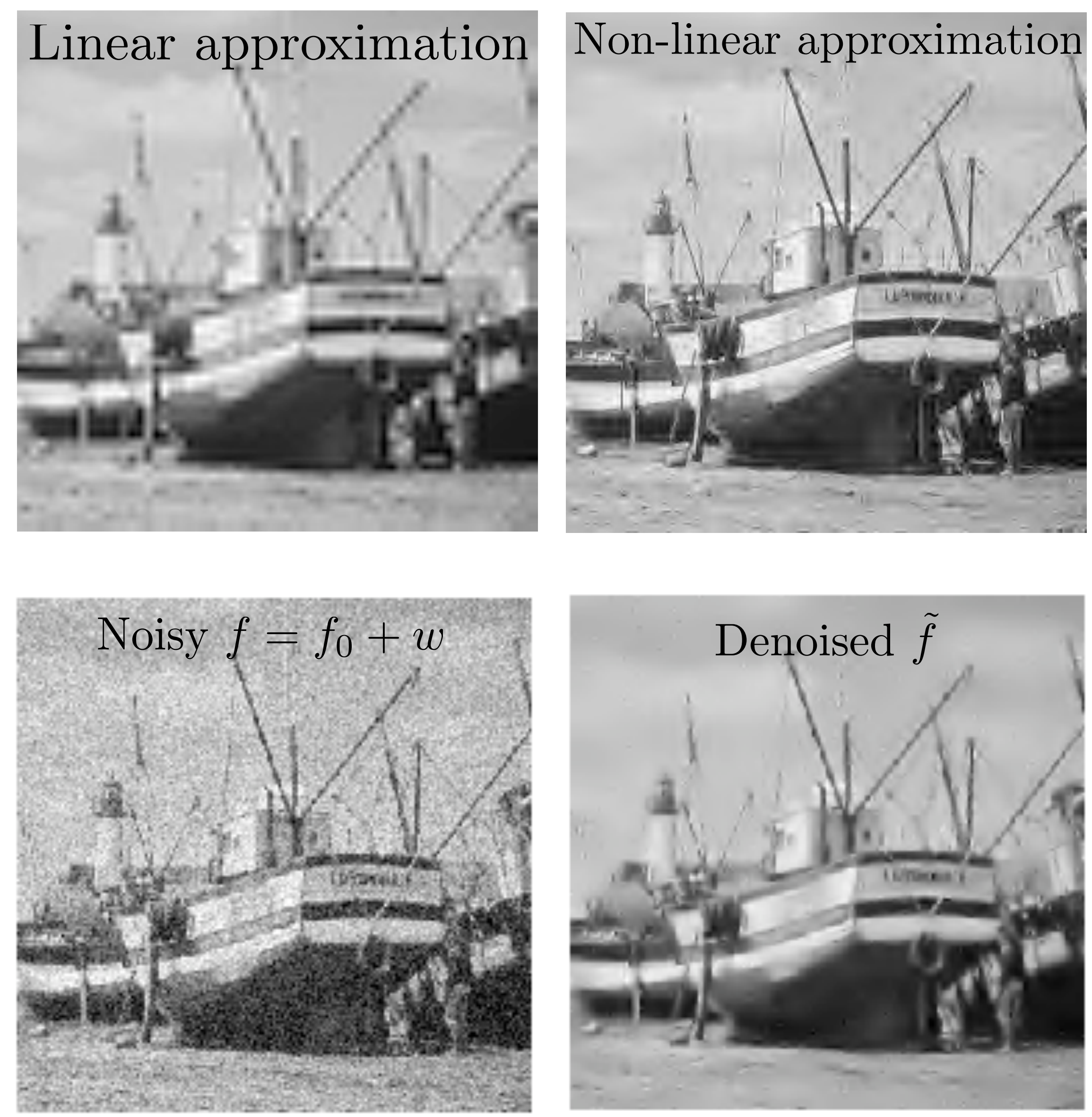
Compression, denoising: ≈ approximation.

Fast decay of approximation error

$\Longrightarrow$ efficient processing.





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Wavelet transform:

Filtering+sub-sampling cascade.